Expectations and Monetary Policy

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Abstract

This paper uncovers a new channel of monetary policy in a New Keynesian DSGE model under the assumption of internal rationality. Agents in the model are fully rational given their imperfect ability to forecast inflation and output. Empirical and policy implications of the model change dramatically under internal rationality relative to rational expectations setting, as the inflation-output trade-off responds to the evolution of agents' expectations. For example, if the Central Banker's main objective is to stabilize inflation, output contracts severely in response to a shock to inflation expectations. Moreover, the Central Bank cannot eliminate shocks to agents' expectations even if the Taylor principle is met. As a result, being tough on inflation can push the economy into a long recession. During the initial phase of a recession agents over-predict inflation, as observed in survey data corresponding to the beginning of the Great Recession. These features would not be observed in the rational expectation version of the model. The paper also contributes to the methodology of the existing learning literature by providing micro foundation for the optimality of the one-step-ahead expectations formation mechanism (adaptive learning) in a standard DSGE model.

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1 Introduction

In the standard *Rational Expectations (RE)* paradigm inflation expectations play limited role in the determination of monetary policy. We show in this paper that inflation expectations can play a fundamental role in determining the adequacy of a certain monetary policy after introducing a minor change to the standard DSGE model. I assume agents do not hold rational expectations, instead, they are *Internally Rational (IR)*, that is they behave rationally given an imperfect view of the economy. Their model of the economy is not exactly correct but it is a reasonable model, validated by the data and survey expectations observations. In this setup a standard Taylor rule that places a large weight on fighting inflation can have very negative effects on output.

Under rational expectations, agents are assumed not only to be rational decision-makers, but also to have a perfect knowledge of the economy and all underlying interdependencies. Models that are studied under RE struggle to simultaneously explain a number of puzzles in the data concerning monetary policy. As is well known, inflation is very persistent, inflation expectations in surveys shows features that are hard to reconcile with RE. Furthermore, Lindé *et al.* (2015) have recently shown that an array of DSGE models cannot explain the persistence of low output during the Great Recession.

Rational expectations is a simplifying assumption that makes it easy to model dynamic economies and to make policy suggestions. At the same time, it is a very strong assumption hence it is worthwhile exploring the effects of deviating from it. Furthermore, it does not provide tools for studying agents' expectations (and corresponding model dynamics) away from a RE equilibrium.

This paper focuses on the expectational channel of monetary policy, which arises when agents' expectations fail to coincide with equilibrium behavior of inflation and output, even though the equilibrium of the model is unique. My approach helps to unveil how unanchored inflation expectations feed back into a real economy and they can modify considerable the dynamics of the economy. I provide evidence favouring the view that this channel has been active in the Great Recession and I also study the role that conventional monetary policy measures play in the trade-off between higher-than-in-equilibrium inflation expectations and output losses.

My starting point is a standard dynamic stochastic general equilibrium (DSGE) model with internally rational economic agents, which are given an almost perfect information structure. The main feature of the model setup is that the agents are equipped with a particular knowledge of the model economy, specifically about the behaviour of inflation and output gap, which differs from the actual processes that hold in equilibrium. Using this knowledge together with the actual data that agents observe, they form their expectations about inflation and output gap for the next period and they make optimal decisions today. This simple deviation from standard RE setting allows for rich dynamics of both expected and realised inflation and output gap in the model, and at the same time, maintains the optimal decision making assumption of all the agents. Although agents knowledge about the economy is imperfect, they hold a “very good” model of equilibrium objects which would not be easy to reject with actual data.

This paper contributes to the *adaptive learning* literature, in that it provides micro foundations for optimality and rationality of one-step-ahead expectations formation in DSGE models under learning. A standard assumption in this strand of the literature is to assume that agents’ expectations are given by an adaptive learning model. In this setting the nature of the resulting decision making process of the agents remains unclear. This paper fills the gap and demonstrates the exact restrictions on information sets of all agents in the economy that together with their beliefs about output gap and inflation result in the exact linearized DSGE model under adaptive learning of Bullard and Mitra (2002).

Furthermore, the previous focus of models of adaptive learning happened to be on the asymptotic properties of the expectations formation mechanism. I restrict my attention to the *short-run* dynamics of economy under learning and focus on the implications for monetary policy. I find that the actual dynamics of the economy differ substantially from those under RE.

One focus of the paper is that this model can explain persistence of low output during a recession. This may be not so surprising, an appealing and well known feature of models of adaptive learning is that they impart more persistence to shocks: since inflation expectations depend on past inflation a shocks today affects many periods in the future. But this persistence alone is not sufficient to explain the persistence of low output during the great recession. I demonstrate that when agents are simultaneously learning about inflation and output gap persistent deviations from RE can happen for a different reason: it turns out that only a particular area of the beliefs space is characterised by an excessive “stickiness” of agents’ beliefs.

I also show that agents beliefs are driven to this area relatively quickly given almost any initial conditions for beliefs. However, once agents beliefs arrive to this region the recovery back to equilibrium slows down considerably and, as a result, the economy may take a very long time to approach an area close to the REE. This happens because this particular region is accompanied by very small forecasting mistakes of the agents which reinforces agents’ beliefs even if they are far from RE equilibrium. This set of conditions is characterised by output gap losses accompanied by inflation expectations being higher than actual inflation. The model produces a strong negative correlation between expectations about inflation and output, which is driven by monetary policy actions. This particular feature of the model behaviour can be used to address the phenomenon of slow recoveries.

The source of the short-run trade-off between inflation and output is the interdependence between their beliefs delivered by the model and monetary actions. The evolution of inflation expectations depends upon the path of expectations about output, and vice versa. It follows that if the central bank does not control the output gap, inflation expectations can easily drift away from the rational expectations equilibrium and the resulting dynamics of economy will depend on model parameters−mainly on the ways in which the central bank acts to stabilize the economy. For some policies there is a large loss in output, accompanied by a high volatility in both inflation and output.

These dynamics can be seen as formalizing the old story of shifting Phillips curves due to changing inflation expectations of the 70s. The short-run implications of learning under IR gives a structured set of tools for analysing these shifts in expectations in a way that they are consistent with optimal behaviour and general equilibrium. At the same time, the New Keynesian IS curve also suggests this trade-off: higher inflation expectations results in lower consumption or output today.

I analyse the policy implications of this expectational channel from two different angles. I perform a comparative statics exercise and study the model behaviour given a single shock to agents' expectations. In this way my approach helps to analyse what happens to both inflation and output if the monetary authorities decide to change an inflation target. Consider a Taylor rule puts a large weight on inflation and low weight on output. The model delivers short-run gains (losses) in output when agents are surprised by a level of inflation that happens to be higher (lower) than expected. If the policy will not react to an overheat of the economy, which is driven by inflation expectations being lower (higher) than the level of realized inflation, output overshoots (tanks) even more before the expectations are driven back to a new equilibrium. However, a policy that does not react to output can be beneficial only if inflation expectations are perfectly anchored to the new RE equilibrium or they happen to be lower than actual inflation. Symmetrically, we find that an economy can be easily pushed into a deep recession if expected inflation is higher than long run inflation for this same Taylor rule. Therefore, the high persistence of low output is more severe when the Taylor rule has a large coefficient on inflation.

I also study optimal Taylor rules in simulations given simultaneous presence of different types of shocks in the model and conclude that optimal policy requires slightly stronger than one-to-one reaction of policy rate to deviations of inflation from its target or, alternatively, aggressive reaction to deviations of both inflation and output gap from their targeted levels.

The remainder of the paper is structured as follows. Section 2 briefly reviews the related literature on learning. Section 3 derives a standard DSGE model under an assumption of internal rationality. Section 4 focuses on the model behaviour under learning. Section 5 explains the existence of the expectational channel of monetary policy and studies the equilibrium path of the model economy given a shock to agents' expectations. This type of setting can be viewed as a change of inflation target by the central bank. Section 5 discusses optimal Taylor rules in simulations and draws policy suggestions. Finally, Section 6 concludes and briefly summarises the main findings of the paper.

2 Related literature

In recent decades the literature developed several approaches that allowed for deviation from the rational expectations hypothesis but at the same time kept the assumption of rationality of economic agents. Some of the methods used to introduce bounded rationality to standard models include rational inattention, private information, Bayesian, and adaptive learning. Woodford (2013) reviews variety of approaches to specifications of expectations of economic agents in dynamic models. This paper considers implications of adaptive learning for monetary policy analysis. Evans and Honkapohja (2001) provide a summary of tools that could be used to analyse models under adaptive learning, with implications for macroeconomics.

The starting point of this paper is the basic version of the New Keynesian sticky-price model of Clarida *et al.* (1999), which is usually solved under an assumption of rational expectations. I am addressing the same model, with the sole modification that agents do not have access to the structure of the model; instead they use observed variables to learn about economic dependencies. The similar setting of this model is studied by Bullard and Mitra (2002); however, they focus on asymptotic properties of learning mechanism, while this paper addresses short-run implications of learning behavior.

Bullard and Mitra (2002) study the expectational stability and determinacy of the resulting equilibrium under the assumption of adaptive learning for a broad class of Taylor rules. They provide central banks with a set of restrictions on Taylor rule coefficients, which guarantees expectational stability and determinacy of the resulting equilibrium. Bullard and Mitra (2007) extend their previous results by considering interest rate inertia. Evans and Honkapohja (2003) demonstrate that interest rate settings, which are optimal under rational expectations, may result in expectational instability once learning is incorporated into the model. They also stress that expectation-based rules, in the case of expectations that are observable to policy-makers, are preferable for guaranteeing e-stability in an optimal interest rate setting. Evans and Honkapohja (2009) provide an extensive review of recent research in this rapidly-evolving field.

This paper is brings the assumption of internal rationality of Adam and Marcet (2011) to otherwise standard DSGE model. Adam *et al.* (2016) match several asset pricing puzzles with a standard consumption-based asset pricing model with internally rational agents who hold subjective beliefs about stock price behaviour. Adam *et al.* (2015) study U.S. stock prices behaviour and demonstrate how asset prices can temporary depart from fundamentals which results in additional volatility in stock prices.

Constant gain learning rapidly gains popularity in adaptive learning literature. The main difference from recursive least squares learning is that it drops the assumption of infinite memory in the expectations formation mechanism of agents. At each point in time it allows for updating agents' perception of parameters, discarding past observations at a constant rate. Marcet and Sargent (1989) demonstrate under which set of conditions this type of learning scheme asymptotically converges to the rational expectations equilibrium.

Recursive updating of beliefs−an expectation formation mechanism under adaptive learning− is supported by empirical literature that has analysed survey data on inflation expectations (Malmendier and Nagel (2016)). At the same time Milani (2007) demonstrates that the standard DSGE model under learning fits the data much better than the same model under REH. Therefore, it is important to study forward-looking macroeconomic models under learning.

3 Basic New Keynesian dynamic stochastic general equilibrium model under Internal Rationality

This section derives the standard New Keynesian dynamic stochastic general equilibrium (DSGE) model in an environment in which agents hold subjective beliefs about the evolution of the aggregate price level and aggregate output in the economy. I demonstrate under which set of conditions the subjective beliefs of internally rational agents are compatible with optimizing behavior even if beliefs differ from probabilities delivered by the model in equilibrium. Additionally, this section provides micro-foundations for the optimality of a one-step-ahead expectations formation mechanism (adaptive learning) in a standard DSGE model, rationalizing beliefs systems in Bullard and Mitra (2002), Evans and Honkapohja (2003), Orphanides and Williams (2005) and others.

The adaptive learning literature usually studies model dynamics, which are produced by the equations that are known to hold for the general equilibrium of the RE version of the same model—but at the same time the objective probabilities in these equations are mechanically substituted with agents' subjective beliefs. The common argument of the adaptive learning literature, which rationalizes the presence of non-model consistent beliefs about inflation and output, is that agents do not have perfect knowledge of the economy; however, this reasoning is broad and does not explain what it is that agents do know in this world. Under this setting it remains unclear to what extent, on a micro-level, agents remain optimal decision-makers and, more importantly, if they are, what is the mechanism that prevents them from discovering the REE of the model and discarding their subjective beliefs for not being model-consistent.

3.1 Households

An economy is populated by a unit mass of identical infinitely-lived consumers. A consumer i chooses her consumption, C_t^i, hours of work, N_t^i, and savings in the form of one-period nominal bond holdings, B_t^i, by maximizing the expected utility subject to a standard flow budget constraint:

$$\max_{\{C_t^i, N_t^i, B_t^i\}_{t=0}^{\infty}} E_0^{\mathcal{P}^h} \sum_{t=0}^{\infty} \delta^t U(C_t^i, N_t^i) \tag{1}$$

$$P_t C_t^i + \frac{1}{1+r_t} B_t^i \leq B_{t-1}^i + W_t N_t^i + D_t \tag{2}$$

$$\underline{B} \leq \frac{B_t^i}{P_t} \leq \overline{B}, \forall t \tag{3}$$

where P_t is the aggregate price level, r_t is the nominal interest rate, W_t is the nominal wage and D_t is a lump-sum component of income. $\underline{B}$ and $\overline{B}$ are finite bounds on real bond holding, introduced in order to avoid Ponzi schemes. The problem defined above is standard to the DSGE literature; the only difference is the expectations operator $E^{\mathcal{P}^h}$.

The expectations of agents are taken with a subjective probability measure $\mathcal{P}^h$, which assigns probabilities to all variables that the agent takes as given when solving her maximization problem. Similar to the standard rational expectations setting of the model, the set of these variables includes the aggregate price level, wages, dividends, aggregate output and the nominal interest rate. However, the main departure from the rational expectations setting is that the probability measure is not necessarily equal to the distribution of those variables delivered by the model in equilibrium. For simplicity I even equip households with true equilibrium mappings of fundamentals of the model into wages, dividends and nominal interest rates. This allows me to focus on the evolution of inflation and output, however, this assumption is still not sufficient for agents to recover rational expectations equilibrium. The household's state space will be formally discussed following the derivations of the general equilibrium of the model. The details about the probability measure $\mathcal{P}^h$ can be found in Section 1.4.

Given that the household is not the price-setting agent in the economy, it seems natural that she holds subjective beliefs about the aggregate price level. A less obvious assumption is that the household i treats the aggregate production level of the final good, Z_t, as an exogenous variable and holds subjective beliefs about its evolution. The aggregate output is consumed by a continuum of households, and from the stance of a single household i her consumption/savings decision does not affect the aggregate level of production. In other words, analogous to the RE setting of the same model, the household i does not internalize the fact that she is a representative agent in the economy. Under the standard assumption of lack of common knowledge, it is impossible for this household to realize that the economy is populated by identical consumers because she does not observe the consumption choice of a continuum of other households in the economy. This is a key assumption that allows a household to hold subjective beliefs about aggregate output, even though in general equilibrium the level of aggregate output will be dictated by a household's choice of consumption level.

Under the assumption of a period utility given by

$$U(C_t^i, N_t^i) = \frac{(C_t^i)^{1-\sigma}}{1-\sigma} - \frac{(N_t^i)^{1+\phi}}{1+\phi} \tag{4}$$

the agent's optimal plan is characterized by the standard first-order conditions:

$$\frac{(N_t^i)^{\phi}}{(C_t^i)^{-\sigma}} = \frac{W_t}{P_t} \tag{5}$$

$$\frac{1}{1+r_t} = \delta E_t^{\mathcal{P}^h}\left[\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{6}$$

Equation (5) defines the consumption/labour choice, and equation (6) is the Euler equation, which pins down the intertemporal consumption decision for household i. These conditions are standard except that expectations are taken with respect to the household's subjective probability measure $\mathcal{P}^h$.

One important assumption is that the household i knows the mapping of aggregate output, Z_t, into wages, dividends and nominal interest rate:

$$\{W_t, r_t, D_t\} = F(Z_t) \tag{7}$$

where F is a true equilibrium relationship which holds in the general equilibrium of the model under learning.[1] Given that the household i internalizes this relationship, she realizes that if real bond holdings are close to zero, her consumption is close to aggregate output.[2] At the same time, she understands that in a given period her consumption is not restricted by the aggregate production level because she is free to choose any level of savings within defined bounds, $[\underline{B}, \overline{B}]$. However, if the bounds for the bond holdings are small enough compared to labour income and transfers, she also internalizes that her expected consumption level for the next period cannot differ much from the aggregate output in the economy. Therefore, combining (7) with the assumption of bounds for real bond holdings being relatively small, the agent can approximate her expected consumption level for the next period using the expected aggregate output. This result is formally stated below:

Proposition 1.

$\forall \widehat{\delta} > 0\ \exists \widehat{\epsilon} > 0$: if $-\underline{B} < \widehat{\epsilon}$ and $\overline{B} < \widehat{\epsilon} \Rightarrow Prob(|E_t^{\mathcal{P}^h}(C_{t+1}^i - Z_{t+1})| < \widehat{\delta}) = 1$.

Proof of Proposition 1: Please refer to Appendix A.

Assumption 1 shows conditions on underlying parameters of the model which guarantee that consumer approximates her consumption for the next period with expected aggregate output in the Euler equation (6).

Assumption 1.

I assume that the bounds for real bond holdings are small, $-\underline{B} < \widehat{\epsilon}$ and $\overline{B} < \widehat{\epsilon}$, so that the result stated in Proposition 1 and the approximation (8) holds with sufficient accuracy (for a given $\widehat{\delta}$).

It follows, that given the Assumption 1 a household i can rely on the following approximation:

$$E_t^{\mathcal{P}^h}\left[\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \simeq E_t^{\mathcal{P}^h}\left[\left(\frac{Z_{t+1}}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{8}$$

Under an Approximation (8) the household's Euler equation (6) takes the following form:

$$\frac{(C_t^i)^{-\sigma}}{1+r_t} = \delta E_t^{\mathcal{P}^h}\left[\frac{(Z_{t+1})^{-\sigma}}{\Pi_{t+1}}\right] \tag{9}$$

[1] In fact, the same relationship also holds in the REE of the model. The details about F are given in Sections 1.3-1.4.

[2] This follows from the first order condition (5) and household's budget constraint (2) combined with (7). Please refer to Appendix A for details.

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the aggregate inflation level. Therefore, the optimal consumption choice is pinned down by agent's expectations of aggregate output and inflation for the next period. Note that under Assumption 1, a labour decision is also close to the rational expectations equilibrium value. This result comes out of the household's first order condition (5) and the budget constraint (2).[3]

The approximation (8) is in the core of the adaptive learning literature which uses one-step-ahead expectations formation mechanisms to formulate agents' beliefs. This approximation rationalizes the "myopic" view of the agent about her consumption decision: to choose the consumption level today she needs to know the expected value of the aggregate output only for the next period. Without this approximation the problem becomes much more complex, because under this setting consumption today depends on future consumption decisions. As a result, the agent would need to take into account long-horizon expectations about the income stream to choose a consumption level today, as in Eusepi and Preston (2011) or as in Adam *et al.* (2015). For simplicity, I study the model under Approximation (8) and this allows me to analyze model dynamics analytically using standard tools.[4]

The argument above shows under what conditions (8) is approximately consistent with optimizing agents. If Assumption (8) holds, household's expectations about future real output play a key role in defining optimal consumption today. In order to claim that the agents are fully internally rational, it is necessary to equip agents with forecasting model that does not lead to systematic misperception in agents beliefs. Therefore, another issue, to be studied separately, will be to see if agents' beliefs imply that agents make easily detectable mistakes while learning. To avoid this problem, I am going to impose additional constraints on agents' model about aggregate output and price level to ensure that their model about these variables is not easily rejected.[5]

3.2 Firms

The single final good in the economy is produced from intermediate goods by perfectly competitive firms, subject to a standard CES production function:

$$Z_t = \left(\int_0^1 Y_t(j)^{1-\frac{1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \tag{10}$$

where Z_t is a quantity of the final good and $Y_t(j)$ is a quantity of the intermediate good j used in

[3]refer to Appendix A for the proof

[4]Adam *et al.* (2016) use a similar simplification in a consumption-based asset pricing model under internal rationality.

[5]Please refer to Section 3.6 for details.

the production of the final good.

The optimality condition in the final good sector leads to the following demand function for the intermediate good j:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Z_t \tag{11}$$

where P_t is a price of the final good (aggregate price index) and $P_t(j)$ stands for the price of the intermediate good j. The aggregate price index is given by $P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$.

The final goods producers are the only agents in the model who are observing the whole distribution of the intermediate goods' prices and know how the price of the final consumption good is formulated. In this model economy their role is limited to transformation of intermediate goods inputs into the consumption good and aggregation of the distribution of the intermediate goods prices into a single price for the final good.

There is a continuum of firms indexed by j in the intermediate goods sector, $j \in [0, 1]$. Each firm produces a differentiated good using the Cobb-Douglas production function:

$$Y_t(j) = A_t N_t(j)^{1-\alpha} \tag{12}$$

where A_t is the technology level (TFP) which is common across all firms and $N_t(j)$ is a labour input.

Producers of intermediate goods operate in an environment with sticky prices. Following Calvo (1983), each firm can reset the price for its good in a given period with a probability $1 - \theta$ and its price remains unchanged with probability θ. The Calvo parameter is assumed to be common across firms in the economy; however any given firm does not realize this.

A firm reoptimizing in period t chooses the reset price, P_t^*, which maximizes the current market value of the profits generated while the price remains effective:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}f} \left[Q_{t,t+k}(P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right] \tag{13}$$

subject to the demand schedule (11) and taking the aggregate output of the final good, Z_t, and the aggregate price level in the economy, P_t, as given. $Y_{t+k|t}$ is the output in period $t + k$ for a firm whose price was last reset in period t, $Q_{t,t+k} = \prod_{j=1}^{k} \delta(Z_{t+j}/Z_{t+j-1})^{-\sigma}(P_{t+j-1}/P_{t+j})$ is the stochastic discount factor for nominal payoffs and $\Psi_t(Y_{t+k|t}) = W_t \left[\frac{Y_{t+k|t}}{A_{t+k}}\right]^{\frac{1}{1-\alpha}}$ is the cost function. The problem defined above is standard; the only difference from RE setting is the expectations operator $\mathcal{P}^f$. Note that firm's expectation operator ($\mathcal{P}^f$) differs from household's one ($\mathcal{P}^h$).

The solution of the profit maximization problem leads to the following expression for the optimal reset price:

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}^f} \left[\delta^k \left(Z_{t+k}\right)^{1-\sigma} P_{t+k}^{\epsilon} MC_{t+k|t}\right]}{\sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}^f} \left[\delta^k \left(Z_{t+k}\right)^{1-\sigma} P_{t+k}^{\epsilon-1}\right]} \tag{14}$$

where the real marginal costs are a known function:

$$MC_{t+k,t} = \frac{1}{1-\alpha} \frac{W_{t+k}}{P_{t+k}} \left[\frac{Z_{t+k}^{\alpha}}{A_{t+k}}\right]^{\frac{1}{1-\alpha}} \left(\frac{P_t^*}{P_{t+k}}\right)^{-\frac{\epsilon\alpha}{1-\alpha}} \tag{15}$$

The optimal reset price depends on the expected evolution of the aggregate price level and aggregate output in the economy. Analogous to the household's problem, under internal rationality the expectations are taken given the firm's subjective probability measure, $\mathcal{P}^f$, while under RE these expectations are defined with a model-delivered (objective) distribution of all exogenous variables. The key feature of $\mathcal{P}^f$ is that its elements are histories of TFP, aggregate output, aggregate prices and wages; the details about the probability measure $\mathcal{P}^f$ are presented in Section 3.4.

Intermediate goods producers are price setters in the economy. However, they do not possess sufficient knowledge to conjecture the relationship between the aggregate price level and their reset price decision in every period. A firm observes only the aggregate price level and does not see the entire distribution of intermediate goods prices in the economy, including the prices of all other intermediate goods producers, which are resetting their prices in the same period. The lack of common knowledge prevents this firm from discovering how their reset price maps into the aggregate price level and therefore allows them to hold subjective beliefs about the aggregate price level.

3.3 General equilibrium and monetary policy

Monetary policy is conducted using a simple Taylor-type interest rate rule. Namely, the central bank sets the nominal interest rate in response to deviations of inflation and output from their targeted levels.

$$1 + r_t = (1 + r^*) \left(\frac{\Pi_t}{1 + \pi^*}\right)^{\varphi_\pi} \left(\frac{Z_t}{Z^*}\right)^{\varphi_z} \tag{16}$$

where π^* is an inflation target chosen by the monetary authorities, Z^* and r^* are the corresponding steady state (natural) levels of output and nominal interest rate.[6] In addition to the inflation target, the

[6]The details about the steady state of the economy can be found in Section B of the Appendix

monetary authorities are also setting the Taylor rule coefficients, φ_π and φ_z, which dictate the degree of reaction of the interest rate to deviations of inflation and output from their targets.

In equilibrium the goods market clears:

$$Z_t = \int_0^1 C_t^i di = C_t^i \tag{17}$$

at all t. Combining the goods market clearing condition with the household's Euler equation (9) leads to the equilibrium condition for the aggregate output:

$$\frac{Z_t^{-\sigma}}{1+r_t} = \delta E_t^{\mathcal{P}^h}\left[\frac{Z_{t+1}^{-\sigma}}{\Pi_{t+1}}\right] \tag{18}$$

Intermediate goods producers are using the same production technology (12) and are facing the same probability of resetting prices for their goods, $1-\theta$. Therefore, all firms that are reoptimizing in a given period are going to choose the same reset price, and the fraction of those firms in the economy equals $1-\theta$. Combining this fact with the definition of the aggregate price level results in the following dynamic of the aggregate price level:

$$P_t = \left[\theta(P_{t-1})^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{19}$$

Market clearing in the labour market requires that

$$\int_0^1 N_t(j)dj = \int_0^1 N_t^i di \tag{20}$$

Combining the labour and goods market clearing conditions with the intermediate firm's production function (12) and the household's optimal consumption/labour relationship (5) leads to the following equilibrium relationships between real wage/dividends and aggregate output:

$$\frac{W_t}{P_t} = Z_t^{\sigma+\frac{\varphi}{1-\alpha}} A^{\frac{\varphi}{\alpha-1}} \tag{21}$$

$$\frac{D_t}{P_t} = Z_t^{\sigma+\frac{\varphi+1}{1-\alpha}}\left(Z_t^{1-\sigma+\frac{\varphi+1}{\alpha-1}} - A^{\frac{\varphi+1}{\alpha-1}}\right) \tag{22}$$

Equations (14) - (22) present the full set of equilibrium conditions that are required to close the model; these conditions describe the evolution of all endogenous variables as a function of fundamentals.

3.4 Further details on agents' state spaces

It was explained above that the information sets of households, final goods producers and intermediate goods producers are not the same in the economy. Given that this is the main feature of the model that prevents agents from discovering the objective probability distributions of aggregate price level and aggregate output, this section reiterates the details about the state spaces of each agent in the economy. More precisely, it explains how the original state spaces of each agent are reduced to the same set of variables for which they share the probability measure, and gives some details about that probability measure.

Households A household i solves her utility maximization problem (1)-(3) taking aggregate production, aggregate price level, wages, dividends and the nominal interest rate as given. It follows that the underlying state space Ω^h consists of the space of realizations of those variables. More precisely, an element $\omega^h \in \Omega^h$ is given by $\omega^h = \{P_t, Z_t, W_t, D_t, r_t\}_{t=0}^{\infty}$ and the household's subjective probability measure $\mathcal{P}^h$ specifies the joint distribution of ω^h. $\{P_t, W_t, D_t, r_t\}_{t=0}^{\infty}$ are in households' state space because they enter the feasibility constraints of consumers; $\{Z_t\}_{t=0}^{\infty}$ appears in the state space because given the knowledge of (7) it determines evolution of $\{W_t, D_t, r_t\}$ in agents' mind and also under Assumption 1 the household approximates her expected consumption level using the aggregate output.

However, to minimize the extent of deviations from the RE version of the model, the household is equipped with the knowledge of several mappings (7), that actually hold in equilibrium under learning,[7] which allow us to reduce its information set to a minimum number of variables in this setting. The first variable which can be removed from the state space is the nominal interest rate. To do so, it is assumed, that the household realizes how the central bank sets the nominal interest rate in the economy. More precisely, the household knows the interest rate rule (16), including the values of the Taylor rule coefficients and levels of inflation and output targeted by the central bank. Additionally, it is assumed, that households are fully aware of the equilibrium mappings of real wages (21) and dividends (22) into the aggregate output and the TFP − a new variable for the households, which needs to be included into a modified state space in this setting. Therefore, (7) is actually given by the joint knowledge of (16), (21) and (22). Given that the households perceive the equilibrium mappings of wages, dividends and the nominal interest rates into the remaining set of exogenous variables, these variables carry only redundant information and without loss of generality can be substituted with the TFP (A_t) in the underlying household's state space.

[7]note that this mappings are identical to those of the rational expectations version of the model

Firms An intermediate goods producer j chooses the optimal reset price for his good by maximizing the expected stream of profits (13), taking aggregate production, aggregate price level, wages and the TFP as given. It follows that the typical element $\omega^f \in \Omega^f$ of the underlying firm's state space Ω^f is given by $\omega^f = \{P_t, Z_t, W_t, A_t\}_{t=0}^{\infty}$ and the firm's subjective probability measure $\mathcal{P}^f$ specifies the joint distribution of ω^f. Note that firm's state space Ω^f differs from household's state space Ω^h.

Similar to that of households, the firm's state space is reduced by equipping it with the exact equilibrium mapping of real wages into the remaining set exogenous variables (21). Therefore, without loss of generality, wages can be excluded from the firm's state space.

All of this means that $\mathcal{P}^h$ and $\mathcal{P}^f$ can be simplified to a probability measure that specifies the joint distribution of histories of the TFP, aggregate output and the aggregate price level. In addition, I assume that households and firms share this probability measure; however, this assumption does not restrict their views about other variables. For example, the firm chooses the dividend, but the household takes the dividend as a given function of fundamentals while solving her maximization problem.

Under this simplification the underlying actual state space of both households and firms Ω consists of the space of realizations of the TFP, the aggregate price level and aggregate output. Specifically, the probability space is given by $(\Omega, \mathcal{P})$ with a typical element $\omega \in \Omega$, $\omega = \{P_t, Z_t, A_t\}_{t=0}^{\infty}$. The probability measure is assumed to be fixed and given by a specific stochastic process, which is close to the actual processes for aggregate inflation and aggregate output, delivered by the model in equilibrium. However, even though agents realize the form of the stochastic process, there are still several parameters (or hidden variables) of the process that are assumed to be unknown to the agents. All the details about this probability measure are given in Section 1.6. Given historical series, jointly produced by the model economy and agents' beliefs about those parameters, households and firms will be updating their perception of those parameters in each period. And this perception is going to influence conditional expectations of future values of aggregate inflation and aggregate output, and therefore affect actual inflation and output today. In this setting, the beliefs about an evolution of the TFP, the aggregate price level and aggregate output form an additional building block for the model, and agents take it as given.

The main difference from the standard RE setting of the same model is that under RE agents' state space consists only of the TFP; however under internal rationality it also includes histories of aggregate inflation and output. The reason why under RE the state space is reduced to the TFP is that it is implied that the agents understand the equilibrium mapping of aggregate inflation and output into the

TFP, and the information contained in those series turns out to be redundant. This is not the case under an assumption of internal rationality, because, as previously explained, if one slightly relaxes the assumption of perfect knowledge it becomes impossible for the agents to discover in one period the policy functions for all endogenous variables in the model. However, note that that rational expectations will arise as a special case of internal rationality for certain choices of belief parameters.

3.5 Log-linearized model

The standard log-linearization of (14) - (21) around zero-inflation steady state delivers the exact model of Bullard and Mitra (2002). Bullard and Mitra (2002) introduced adaptive learning in the standard DSGE model by simply plugging in subjective beliefs into equilibrium relationships, which are known to hold in the RE version of the model. The analysis performed above disentangles the exact requirements for the information sets of all agents, which allow them to internally consistently hold subjective beliefs about aggregate variables in the economy. Further, this analysis argues that in this environment it is impossible for agents to easily discover the REE of the model. Additionally, it confirmed that on the micro level all agents remain optimal decision-makers that have their own view of the economy, as summarized by $\mathcal{P}$.

The model, log-linearized around the zero-inflation stead state, consists of three standard blocks: the dynamic IS, the Phillips curve and the Taylor rule.

$$z_t = E_t^{\mathcal{P}} z_{t+1} - \sigma^{-1}(r_t - r_t^n - E_t^{\mathcal{P}} \pi_{t+1}) + \epsilon_t^z \tag{23}$$

$$\pi_t = \kappa z_t + \delta E_t^{\mathcal{P}} \pi_{t+1} + \epsilon_t^{\pi} \tag{24}$$

$$r_t = \varphi_\pi \pi_t + \varphi_z z_t \tag{25}$$

where z_t is the output gap, π_t is inflation and π_{t+1}^e and z_{t+1}^e are their one-period-ahead expectations. r_t is the monetary policy instrument/nominal interest rate. σ, κ, δ are micro-founded parameters of the sticky price model; the notation is standard for the literature and the exact expressions are presented in Appendix B. The model can be extended to the stochastic environment; therefore demand cost push shocks, ϵ_t^{π} and ϵ_t^z, can be incorporated into the basic setting.[8]

An assumption that the TFP is the first-order autoregressive process pins down the same properties for the process for the natural rate of interest, r_t^n, which is a scaled measure of the TFP, as explained

[8]Note that this extension requires modification of state spaces of each agent in the model.

in Appendix B. Therefore, under this assumption the natural rate of interest also follows the first-order autoregressive process, as in Bullard and Mitra (2002):

$$r_t^n = \rho r_{t-1}^n + \epsilon_t \tag{26}$$

where ρ is the parameter of an autoregressive model. The main stochastic disturbance in the model is given by an exogenous shock to the natural rate of interest, ϵ_t, which is assumed to be normally distributed with $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The knowledge of r_t^n provides to the agent information about the evolution of the technology in the economy and without loss of generality it is possible to substitute TFP with r_t^n in the agents' state space.

The system of equations (23)-(25) can be summarised in the following way:

$$y_t = B y_{t+1}^e + \chi r_t^n + B_1 \epsilon_t^y \tag{27}$$

where $y_t = \begin{bmatrix} z_t \\ \pi_t \end{bmatrix}$, $\epsilon_t^y = \begin{bmatrix} \epsilon_t^z \\ \epsilon_t^\pi \end{bmatrix}$, $B = \frac{1}{\sigma+\varphi_z+\kappa\phi_\pi} \begin{bmatrix} \sigma & 1-\delta\varphi_\pi \\ \kappa\sigma & \kappa+\delta(\sigma+\varphi_z) \end{bmatrix}$,
$B_1 = \frac{1}{\sigma+\varphi_z+\kappa\varphi_\pi} \begin{bmatrix} \sigma & -\varphi_\pi \\ \kappa\sigma & \sigma+\varphi_z \end{bmatrix}$, and $\chi = \frac{1}{\sigma+\varphi_z+\kappa\varphi_\pi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix}$.

Under the concept of *rational expectations*, the agents have full knowledge of the structure and all interdependencies in a given economy. The model (27) is solved by forward iterations, and the rational expectations equilibrium (REE) is given by:

$$y_t' = (1, r_t^n)\overline{\beta} + B_1 \epsilon_t^y \tag{28}$$

where $\beta = \begin{pmatrix} a^z & a^\pi \\ c^z & c^\pi \end{pmatrix}$ denotes a generic vector of coefficients and in the REE $\overline{\beta}$: $(\bar{a^z}, \bar{a^\pi})' = 0$ and $(\bar{c^z}, \bar{c^\pi})' = (I-\rho B)^{-1}\chi$. Where, for example, a^π is the intercept in the policy function for inflation and c^π is a slope coefficient that explains how the dynamic of inflation is dependent upon the dynamic of the natural rate. Therefore, under rational expectations the evolution of the economy is fully described by (26) and (28).

Under internal rationality, it is impossible for agents to discover the objective probability distribution of inflation and the output gap, summarized by (26) and (28) under RE. To close the model (26)-(27) it is necessary to specify agents' perception of the dynamics of inflation and the output gap in this economy, which in this setting is an additional building block of the model. This perception, together with observable historical data, makes it possible for agents to form expectations about inflation and output for tomorrow. And agents are actually interested in making precise forecasts of aggregate inflation and aggregate output gap, given all available information in each period, because their consumption/price setting decisions today are dependent upon their evolution. For example, if the inflation expectations differ substantially from realized inflation it implies that the choices agents made conditional on the expected value are no longer optimal and might be distant from a revised optimum, given a new, extended information set.

Even though the agents' perceived probabilities about ω are not going to be equal to the equilibrium distribution of inflation and output, I choose agents' perception in such a way so that the agents are not making obvious mistakes while learning. However, it should be noted that given that the agents are learning about endogenous variables, small forecasting mistakes does not necessarily imply that the expectations are close to rational expectations equilibrium. Interestingly, even small deviations in agents' perception from the objective distribution of inflation and output might generate significant changes in the actual dynamics of both variables under learning compared to those predicted by the REE version of the same model, and the recommendations for monetary policy are going to differ substantially. The details about agents' probability measures and optimal updating of beliefs are summarized in the next section.

3.6 Probability measure $\mathcal{P}$ and optimal beliefs updating

The probability measure $\mathcal{P}$, which is the same for both households and intermediate goods producers, specifies the joint distribution of the TFP/natural rate of interest, aggregate inflation and aggregate output gap.

I am assuming that there is no misspecification in agents' perception of the process for the natural rate of interest/TFP. Namely, both households and firms understand (26) and know the exact values of all characteristics of this process. Consequently, all agents in the model economy hold rational expectations about the natural rate of interest.

The main deviation from the rational expectations version of the model is agents' view about an evolution of inflation and output gap. Even though the agents' perception of the process is going to differ from the process delivered by the model in equilibrium (28), the extent of this deviation is kept to a minimum in a sense that I will explain in details below. Namely, even though the agents do not know the exact values of coefficients in (28), they understand that inflation and output are driven by the natural rate of interest. The perceived law of motion for this model takes the following form:

$$y_t = (1, r_t^n)\beta_t + \zeta_t \tag{29}$$

$$\beta_t = \beta_{t-1} + \xi_t \tag{30}$$

for $\zeta_t \sim N(0_{2,1}, \sigma_\zeta^2 * \iota_2)$, $\xi_t \sim N(0_{2,2}, \sigma_\xi^2 * I_2)$ independent of each other and an evolution of r_t^n is described by (26). In other words, the agents understand the functional form of equilibrium laws of motion for inflation and output gap; however they are uncertain about the coefficients of that function. Agents' view of the process for inflation and output gap is more general than (28) and allows for changes in the relationship between these variables and the natural rate of interest.

Note that this specification for the agents' beliefs about evolution of inflation and output in the economy encompasses the REE of the model. If agents believe that $\sigma_\xi^2 = 0$ and they have a prior belief that $\beta_0 = \overline{\beta}$ with certainty, it holds that the resulting equilibrium is indeed REE.

The process (29) is a standard unobserved component model, and the learning problem arises because agents only observe realizations of y_t and r_t^n and not the permanent and transitory components of the policy function coefficients that relate them to each other. The optimal filter for this type of process is a Kalman filter, which filters out the noise component ζ_t and at the same time keeps track of permanent innovations ξ_t.

For simplicity, assume that agents' prior beliefs are centered at the REE:

$$\beta_0 \sim N(\overline{\beta}, \sigma_0^2) \tag{31}$$

and fixing σ_0^2 to be equal to the steady state Kalman filter uncertainty about β_t. Agents' posterior beliefs are given by $\beta_t \sim N(\widehat{\beta}_t, \sigma_0^2)$ and it is well known that optimal updating implies the following recursion, which explains the evolution of agents' beliefs about β_t:

$$\widehat{\beta}_t - \widehat{\beta}_{t-1} = \lambda_t R_t^{-1} \begin{bmatrix} 1 \\ r_t^n \end{bmatrix} \left[y_t - \begin{pmatrix} 1, & r_t^n \end{pmatrix} \widehat{\beta}_{t-1} \right] \tag{32}$$

$$R_t - R_{t-1} = \lambda_t \left[\begin{bmatrix} 1 \\ r_i^n \end{bmatrix} \begin{pmatrix} 1, & r_i^n \end{pmatrix} - R_{t-1} \right] \tag{33}$$

where $\widehat{\beta}_t$ is agent's estimate of β_t in (29). λ_t defines a weight that agents put on the most recent observation to update their forecasts from period $t-1$ to a period t. This is the optimal Kalman gain and its value is defined by the signal-to-noise ratio, q (ratio of variances of permanent and transitory components $q = \frac{\sigma_\zeta^2}{\sigma_\xi^2}$, which captures the informativeness of the last observation). Therefore, the constant gain is the known function of the signal-to-noise ratio:

$$\lambda^* = \frac{q + \sqrt{q^2 + 4q}}{2 + q + \sqrt{q^2 + 4q}} \tag{34}$$

for $q > 0$ and for the case of $q = 0$ the weight on the last observation is $\lambda_t = \frac{1}{t}$. The last case is a recursive least squares learning, and at each point in time, t, agents will update their estimates of coefficients, adding new observations with the weight $1/t$. Under this condition, (32)-(33) transform into the exact expectations formation mechanism of Bullard and Mitra (2002). However, if the agent believes that there is some variation in the permanent component, $q > 0$, the optimal filtering in this environment would give rise to constant gain learning. This implies that instead of estimating coefficients in (29) by running basic OLS, constant gain learning assumes that agents are using exponentially-weighted least squares to form their expectations. This setting allows agents to slowly discard past observations while incorporating each new observation with the same weight $-$ constant gain.

3.7 Equilibrium dynamics under learning

Applying the expectation formation mechanism (29) to the model economy (27), one can derive the actual law of motion for inflation and output gap. Agents' expectations about inflation and output gap are defined according to (29) and given by:

$$E_t^{\mathcal{P}} y_{t+1} = \hat{a}_t + \hat{c}_t \rho r_t^n \tag{35}$$

where $\hat{a}_t$ and $\hat{c}_t$ are optimal estimates of β_t in (29). It follows that the actual dynamics of output gap and inflation are a function of agents' beliefs about those variables:[9]

$$y_t = Ba + (Bc\rho + \chi) r_t^n + noise_t \tag{36}$$

Following the methodology of Evans and Honkapohja (2001), using the actual law of motion (36) and the perceived law of motion (29), one could formulate the function that maps agents' expectations of parameters to their realized values. For example, inflation today depends on what agents believe it is going to be tomorrow; at the same time, what agents believe inflation is going to be tomorrow

[9]this can be easily verified by substituting (35) into (27)

depends on its current value. So, in this simple self-referential system the actual value of the observable parameter, β, depends on agents' forecasts of β through the known functional form $T(\beta)$, which is recovered from the structural model and is not known to agents. This function is called a T-mapping of the learning scheme and for the case of this model is given by:

$$T(a, c) = (Ba, Bc\rho + \chi) \tag{37}$$

In other words, if agents expect the predictable part of endogenous variables to be $y_t = a + cr_t^n$ the actual deterministic part of output gap and inflation is given by $y_t = T_a(a, c) + T_c(a, c)r_t^n$. A fixed point of this mapping, a solution to $a = T_a(a, c)$ and $c = T_c(a, c)$, is a REE of the model (28). The T-mapping defines the expectational channel in this model economy and allows us to study an evolution of beliefs on the way to equilibrium. By incorporating the knowledge of the T-mapping, the scheme for sequential updating of beliefs (32) in the model economy can be rewritten in the following form:

$$\widehat{\beta}_t - \widehat{\beta}_{t-1} = \lambda_t R_t^{-1} \begin{bmatrix} 1 \\ r_t^n \end{bmatrix} \left[\begin{pmatrix} 1, & r_t^n \end{pmatrix} (T(\widehat{\beta}_{t-1}) - \widehat{\beta}_{t-1}) + noise_t \right] \tag{38}$$

$$R_t - R_{t-1} = \lambda_t \left[\begin{bmatrix} 1 \\ r_i^n \end{bmatrix} \begin{pmatrix} 1, & r_i^n \end{pmatrix} - R_{t-1} \right] \tag{39}$$

where $T(\beta)$ is given by (37). In this paper I am considering both cases of recursive least square learning ($\lambda_t = \frac{1}{t}$), as in Bullard and Mitra (2002), and constant-gain learning.[10]

Interestingly, the optimal filtering dictates that agents' expectations are only partially updated with each new observation, therefore learning generates high persistence in agents' beliefs. It also follows that persistence in beliefs under learning, equation (38), will transmit to the persistence in actual variables. Implicitly, through inflation expectations, the actual level of inflation will be a function of past realizations of inflation $-$ this channel is absent if one studies this DSGE model under REH, but is crucial for this type of model in order to fit the data.[11]

Marcet and Sargent (1989) demonstrate that under RLS learning as $t \to \infty$ the system (38)-(39) follows an ordinary differential equation (O.D.E.) and converges to REE if the conditions of stability of

[10]For the latest case I assume $\lambda = 0.02$ $-$ the value recovered by several empirical studies.

[11]There is a known issue that this simple DSGE model under REH generates very low persistence in inflation series an this finding contradicts empirical studies. At the same time Milani (2007) demonstrates that a standard DSGE under an assumption of constant gain learning generates persistence in inflation which fits US data.

O.D.E. are met. An approximation for dynamics of beliefs away from REE is given by a small O.D.E. that assumes that (39) is in equilibrium:

$$\dot{\beta} = T(\beta) - \beta \tag{40}$$

Williams (2014) proves that even for a case of constant gain learning, the same O.D.E. (40) describes the evolution of beliefs in the short run if λ is small.[12] And this short-run dynamic is the main focus of this paper.[13]

4 Model behaviour under learning

4.1 Standard approach to monetary policy design: determinacy under REE and expectational stability under learning

This section summarizes the analysis of Bullard and Mitra (2002) to provide an overview of the standard approach to conducting a desirable monetary policy under rational expectations and learning.

Under rational expectations the monetary policy rule should be designed in such a way that the resulting equilibrium is unique. For the case of the standard DSGE model outlined above, the determinacy of the REE (28) is guaranteed if all eigenvalues of matrix B in (27) lie inside the unit circle.[14]

The issue that arises under learning is if the monetary policy rule is designed in such a way that it ensures that if agents' beliefs are away from REE, those beliefs quickly return to equilibrium. The unique stable point of the model under learning is REE, and if the beliefs happen to be initialized away from the equilibrium, the consecutive path of those beliefs is given by (38)-(39). The equilibrium is defined as expectationally stable (e-stable) if, once initialized away from REE, beliefs converge back to their REE values. This happens if the conditions of stability of (40) are met: the real part of all eigenvalues of the first derivatives of $T(a, c)$ with respect to a and c are smaller than one.

The e-stability of the learning scheme in this model economy also sets conditions on the eigenvalues of matrix B; however, these conditions are in principle different from those that guarantee the determinacy

[12] please refer to Theorem 4.3 in Williams (2014) for details

[13] the value or functional form of the weight the agents put on the last observation, λ_t in (38), does not change the path of agents' beliefs; however, it does define the speed at which those beliefs move along the same path

[14] Blanchard and Kahn (1980)

of REE.[15] The source of this difference stems from the principle that the determinacy of REE requires stability of the linear difference system (27), although stability of beliefs under learning depends on the stability of the linear differential system (40). However, for the case of this particular model, if Taylor rule coefficients satisfy (41), the resulting equilibrium is unique under REH and e-stable under learning.[16]

$$\kappa(\varphi_\pi - 1) + (1-\delta)\varphi_z > 0 \tag{41}$$

And the rate at which the successive iterates of (38) approach the fixed point − rate of asymptotic convergence, − is defined by the linear approximation of the T-map around the fixed point.

In order to visualize these results and define a preferable set of Taylor rule coefficients according to the criteria of the value of the largest eigenvalue, I am applying the calibration for the US economy used by Bullard and Mitra (2002) to the standard model outlined above. It is assumed that $\sigma = 0.157$, $\kappa = 0.024$ and $\delta = 0.99$ − the values of the parameters that are considered plausible in business cycle literature. Taylor rule coefficients for a baseline case are set to standard values, $\varphi_\pi = 1.5$ and $\varphi_z = 0.5$, as proposed by Taylor(1993).

Matrix B is a two-by-two matrix, therefore in a general case it has two distinct eigenvalues. From now on I am using the following notation: h_1 and h_2 are the eigenvalues of B, assume that $h_1 > h_2$. v_i stands for an eigenvector and V_i for an eigenspace/eigenline associated with the eigenvalue h_i, $i = 1, 2$.

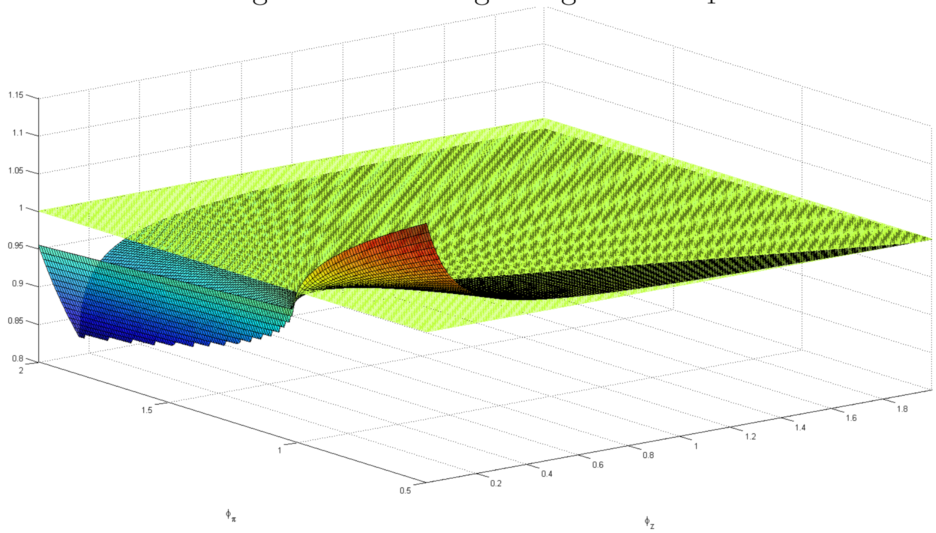
Given the calibration stated above, it is easy to confirm that policy rule coefficients satisfy the condition stated in (41) and, therefore, a resulting REE is unique and expectationally stable. To demonstrate the whole set of Taylor rule coefficients that guarantees e-stability, for every pair of policy coefficients from a feasible set, the largest eigenvalue of matrix B (h_1) is calculated and the resulting surface is plotted on Figure 1.

The light green plane on Figure 1 marks a unitary level to highlight the whole region of e-stability: the entire area of policy coefficients that is below this plane satisfies the e-stability principle. It follows that the Taylor rule, which is active with respect to deviations of inflation from the targeted level, guarantees that even if beliefs are not anchored at the targeted level they will eventually catch up with REE values. The analysis of Bullard and Mitra (2002) finishes here as it focuses only on e-stability of beliefs under learning.

[15]The first derivatives of $T(a, c)$ are given by $[B, \rho B]$. So, given that $\rho < 1$, the largest eigenvalue will be among eigenvalues of matrix B.

[16]please refer to Appendixes A and B in Bullard and Mitra (2002) for the proof

Figure 1: The largest eigenvalue h_1



4.2 Speed of convergence of beliefs to REE

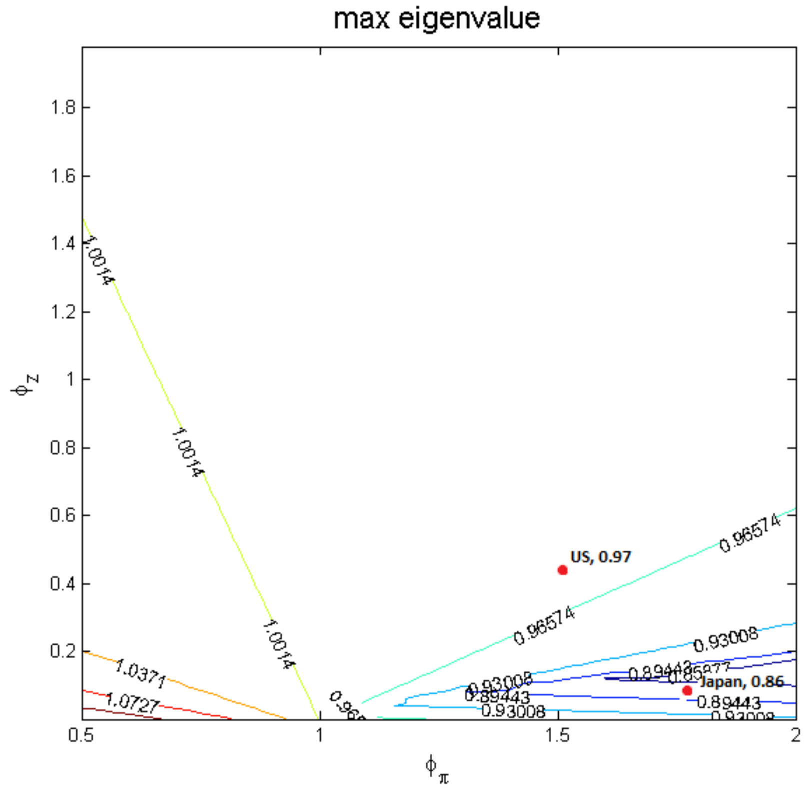
It is also straightforward to refine the results of Bullard and Mitra (2002) and define the set of Taylor rule coefficients that guarantees not only the e-stability of agents' beliefs, but also a faster rate of convergence to REE.

Figure 1 gives a sense of which set of Taylor rule coefficients − the dark blue area of the surface in this case − leads to smaller values h_1, and demonstrates the smallest value of h_1 that could be achieved given the feasible set of Taylor rule coefficients.

Figure 2 presents some of the information contained in Figure 1 in a two-dimensional plane. In this figure the contours of the h_1-surface are plotted in the same space of Taylor rule coefficients as in Figure 1. The blue area on the graph indicates a set of policy coefficients, which ensures not only determinacy and e-stability, but also a higher speed of convergence. This policy dictates an aggressive reaction to deviations of inflation from the target, accompanied by a minor reaction to output gap deviations. This policy ensures that even if beliefs happen to disagree with their REE values, they would return to equilibrium as fast as possible given the constraints for policy coefficients.

While setting the Taylor rule coefficients, apart from simply influencing the determinacy and e-stability of economy, monetary authorities are changing the general interdependence of expectations about inflation and output gap. So, a monetary policy that attempts to anchor agents' expectations should take into account several compromises while setting its interest rate. In this model, even given that the properties that guarantee the asymptotical convergence to REE are met, the actual path of

Figure 2: h_1−contour



beliefs that is pinned down by that equilibrium might also matter.

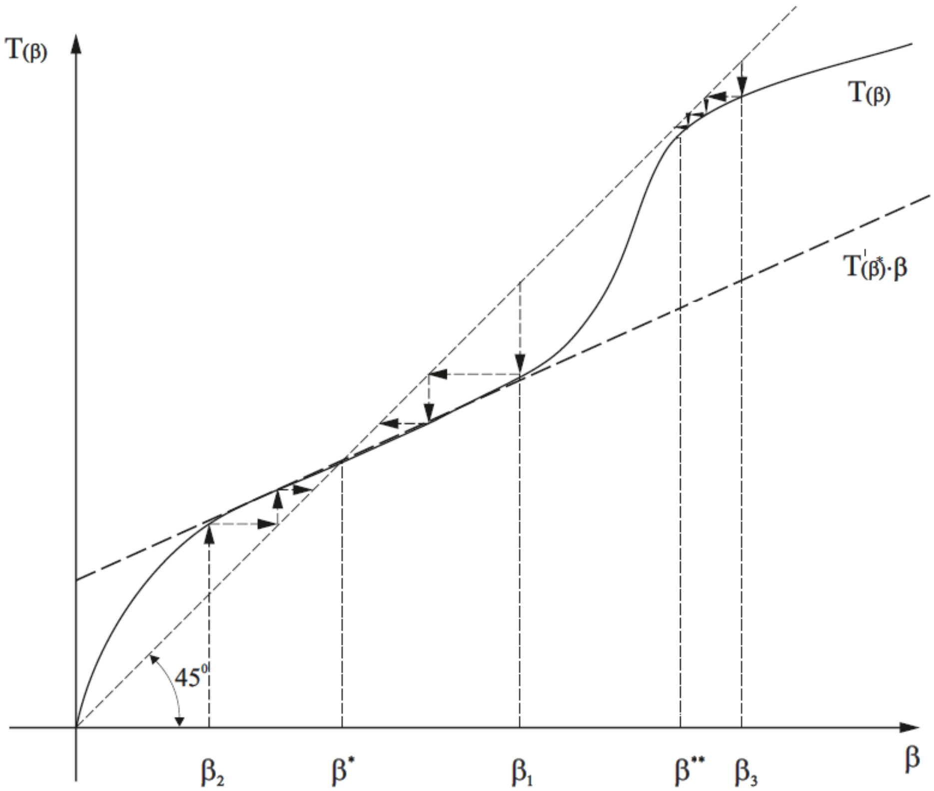
4.3 Actual dynamics of beliefs away from the REE

Most of the existing learning literature focuses on issues of the expectational stability of the learning scheme. It follows that the main focus is the largest eigenvalue of the T-map, linearized around REE, and the lower value of this eigenvalue ensures faster speed of convergence of beliefs to the REE. One main point of this paper is that only if agents are learning about a single parameter, and that the underlying T-map is linear with respect to that parameter, does the eigenvalue describe the entire evolution of beliefs on the way to REE. If one of those conditions is not satisfied and the beliefs are initialized away from the neighbourhood of the equilibrium, the resulting dynamic of the learning mechanism cannot be captured by a single eigenvalue and will instead be driven by other characteristics. More generally, the model dynamic might be very different from the evolution of beliefs predicted by asymptotic properties that hold in the neighbourhood of REE. In this case, therefore, the largest eigenvalue/local speed of convergence might not be the primary concern for policy suggestions.

To demonstrate the first issue - non-linear T-mapping, as in for example Eusepi and Preston (2011)

- Figure 3 illustrates the connection between the shape of T-mapping and the speed of convergence to the REE for a simple one-dimensional case in which agents learn about a single coefficient β.

Figure 3: Mapping from perceived to actual law of motion



The actual law of motion for the parameter of interest, β, depends on agents' perception of it. The curve of the Figure 3 is an arbitrary non-linear T-mapping with a single stable point - β^* - which represents the REE for this example. The agents do not know the equilibrium value, β^*, and they use available data to recover it. In order to incorporate new observations they use either recursive least squares or constant gain learning to re-evaluate and correct their estimates of β from period to period.

The dashed line on Figure 3 stands for the linear approximation of T-mapping at the REE β^*. The speed of local convergence is characterised by the slope of the linearized T-mapping at the REE: the steeper the slope (the lower the eigenvalue) the slower the local convergence. This result is intuitive, because the flatter slope of the T-map results in a larger forecasting mistake for a given belief, and, optimal filtering often imples that agents will revise their expectations with the forecasting error, those beliefs would be pushed closer to the REE.

For a case in which T-mapping is one-dimensional and is linear in β, the local approximation corresponds to the actual shape of the T-map and therefore describes not only local but also global properties of the learning scheme. However, if T-mapping is non-linear, as in the case of Figure 3, this result no longer holds, and the difference between the linearized and actual evolution of beliefs might be substantial. In this case, in order to draw policy suggestions, it is particularly important to understand the type of initial conditions or disturbances to beliefs the economy is facing. For example, if the system starts at value β_1 or β_2, the local speed of convergence is an accurate approximation of global convergence, but if the system is initialized at β_3, the actual dynamic of the system would be very different from the one predicted by the linearization.

The example of Figure 3 illustrates the T-map with an additional approximately self-confirming equilibrium - β^{**}. And therefore, the learning scheme contains the possibility of becoming stuck at β^{**} for a long time, despite the fact that β^{*} is the only stable point in this case. In the neighbourhood of β^{**} the forecastable part of the agents' mistake $(T(\beta^{**}) - \beta^{**})$ in the estimation of β might be so small compared to the exogenous shocks that it will take a long time for those agents to update their estimates in the direction of equilibrium. If one considers an example with inflation expectations, the non-linearity of the T-map might lead to a self-confirming equilibrium with higher-than-in-REE inflation, even though the conditions of e-stability are met and expectations and actual inflation will eventually return to the neighbourhood of REE.

The important observation to take from this simple example is that the policy suggestions might be very different for the cases if one looks at the learning scheme with or without linearization. This actually leads to the question of whether the focus should be entirely on the neighbourhood of the fixed point, or whether larger deviations of beliefs from equilibrium values may also be of interest. If one wants to design a policy in such a way that, once initialized away from the REE, beliefs return to the REE neighbourhood as fast as possible, the local speed of convergence/eigenvalue of the T-map might not be the appropriate measure for the case of arbitrary initial conditions. The entire shape of the T-map and its transformation with changes in policy-controlled parameters should be taken into account for the policy design. The trade-offs between the slope of the T-map around the REE and the "stickiness"/size of the area around the temporary equilibrium, as presented by the neighbourhood of β^{**} on Figure 3, can easily arise in this environment.

However, issues of non-linearity are not the focus of this paper. Instead we focus on the issue of multidimensionality of T-mapping. If agents are simultaneously learning about the evolution of two variables, in the general case their beliefs about those variables are going to be interconnected.

The model outlined in the previous section is an example of a linear, four-dimensional T-map. For simplicity, assume that there is no variation in the natural rate of interest, and therefore agents are learning only about constant terms for inflation and output gap in policy functions (29), so $c_t = 0$, $\forall t$, in an environment without any additional disturbances. More precisely, consider the following dynamic system which describes an evolution of beliefs:

$$\beta_t = B\beta_{t-1} \tag{42}$$

where $\beta_t = [a_t^\pi, a_t^z]'$ and B is for the purpose of this section an arbitrary matrix with both eigenvalues between 0 and 1: $0 < h_2 < h_1 < 1$. β_0 is an arbitrary initial condition for beliefs and we are interested in the consequent path that the economy will take given this initial condition for beliefs. Note that (42) is just a simplification of (38) under the described set conditions. As in a single-dimensional case, the largest eigenvalue of matrix B (h_1) is responsible for the number of time periods during which the system stays away from the neighbourhood of REE, given the initial condition.[17]

One important characteristic for the path that the dynamic system (42) takes on the way to equilibrium is the eigenvector (v_1) that is associated with the largest eigenvalue (h_1). The slope of the corresponding eigenline V_1 defines the asymptotic relationship between the elements of the dynamic system: the long-run relationship between beliefs about inflation and output gap for the case of this model. If the economy happens to satisfy a condition in which h_1 is close to 1 and h_2 is small, the system will find itself on the eigenvector v_1 almost straight away for virtually any initial condition for beliefs. The only exception would be when the initial condition belongs to V_2.

Formally for an arbitrary initial condition in the short run, the dynamic of beliefs will be dictated by both eigenvalues and eigenvectors. However, along the eigenvector v_2 that corresponds to the lowest eigenvalue, the beliefs move with the speed of the smaller eigenvalue h_2 and, therefore, faster in the direction of the other eigenvector v_1. This implies, that while the evolution of beliefs is dominated by that eigenvalue, agents are making large forecasting mistakes while learning. And, given that they update their expectations with the forecasting mistake each period, it follows that agents will quickly discard this region of beliefs. However, once the system reaches v_1, the movement slows. This leads to smaller forecasting mistakes in the process of learning because from this moment the evolution of beliefs is mostly controlled by the largest eigenvalue h_1. It follows that if h_1 is close to one, agents' beliefs will be close to the actual realizations of variables and, therefore, will become stuck once they hit the eigenline V_1.

[17] Details regarding eigenvalue representation of the dynamic system are covered in Appendix B.

In this set of conditions a significant decrease in forecasting mistakes does not imply that agents' beliefs reached the neighbourhood of the REE. In an environment in which agents are simultaneously learning about values of two endogenous variables this implies that beliefs have become aligned with the eigenvector, corresponding to the largest eigenvalue of the T-map. And, this might happen at a sufficient distance from equilibrium along the corresponding eigenline. If the difference between two eigenvalues is large, the effect of the smallest eigenvalue h_2 does not persist, but the transition dynamics on the way back to the equilibrium distribution of beliefs might have striking policy implications.

If learning is considered in the stochastic environment, the dynamic system takes the following form:

$$\beta_t = B\beta_{t-1} + \epsilon_t \tag{43}$$

where ϵ_t is a two-dimensional vector of i.i.d shocks and $\epsilon_t \sim N(0, \sigma_\epsilon^2 * I)$. The system (43) describes an evolution of beliefs in (38) with an assumption of exogenous shocks to inflation and output gap.

For the case of specification for the law of motion of beliefs (43) it follows that their limiting distribution $-$ the covariance matrix of β_t as $t \to \infty$ $-$ is also dependent upon the matrix B and therefore its eigenvalue representation. More precisely, if h_1 is close to one and h_2 is small, the ergodic distribution of beliefs will be stretched along the eigenvector associated with h_1. The closer the eigenvalue is to one, the more stretched will be the covariance ellipse along this direction.[18] This implies that given any variance of independent disturbances to expectations about inflation and output gap, ϵ_t, the actual inflation and output gap in this economy is going to be located in the neighbourhood of the corresponding eigenspace and the higher value of the eigenvalue, with a high probability that these beliefs can be pushed far away from the REE along the corresponding eigenline.

For the case of the model described in this paper, the eigenvector v_1 associated to the largest eigenvalue h_1 will describe both the long- and medium-run trade-off in inflation-output expectations under learning if beliefs are initialized away from the equilibrium. In this model it also follows, that an actual inflation and output gap evolve according to the same dynamic system (37). And therefore, the slope of the eigenline V_1 will also define real output costs of inflation expectations being higher than in REE. Consequently, a policy that simply targets the value of h_1, implying that it speeds up beliefs regarding the slowest variable in the model in the direction of equilibrium, might result in high output losses (steeper V_1) for given higher-than-in-REE inflation expectations. What should be taken into account while choosing the policy suggestion is the actual path that this policy sets for beliefs and,

[18]Please refer to Appendix C for details.

as a result, for the entire economy on the way to achieving equilibrium. It might be the case that a slower movement of beliefs in the direction of equilibrium is preferable to large, amplified oscillations on the way to equilibrium. The extent of the inflation-output trade-off in this simple model is covered in the next section.

5 Expectational channel of Monetary Policy

This section applies the analysis described above in terms of an arbitrary stable dynamic system for the case of the New Keynesian DSGE model when agents are simultaneously forming their expectations about inflation and output gap. Agents' expectations about inflation influence the path for the real side of the economy in this setting. This environment creates an *expectational channel of monetary policy* because it turns out that the evolution of beliefs and their interdependence are highly dependent upon the values of Taylor rule coefficients. So, the behaviour of an economy that comes from the evolution of agents' beliefs away from REE and its transformation with changes in Taylor rule coefficients is considered to be an expectational channel of monetary policy, and therefore should be analysed in addition to standard dynamics of inflation and output gap in the REE.

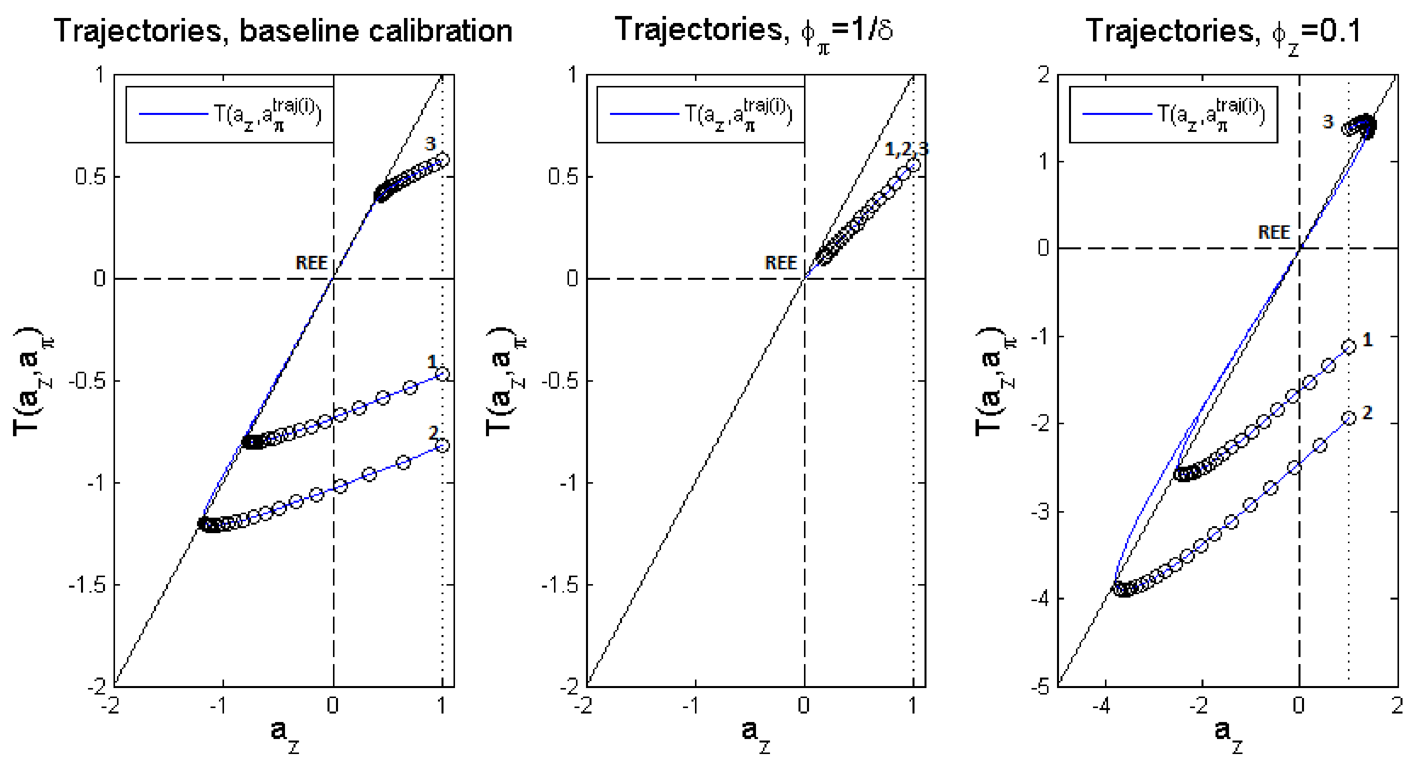
The expectational channel I am describing in this section does not arise due to a problem of multiplicity of equilibria in a standard DSGE model. In an environment with a unique equilibrium, agents' beliefs being away from the REE brings about an abnormal dynamic in the actual series for inflation and output gap if compared to their corresponding paths in the REE.

5.1 Interdependence of learning behaviour

The key characteristic of the standard model outlined in Section 3, which gives rise to the expectational channel, is that agents are simultaneously learning about two endogenous variables: inflation and output gap. Given the way that monetary policy is conducted (25) in the model economy described above, and given that inflation and output gap depend on the natural rate of interest (27), the model delivers a fundamental correlation between these variables under rational expectations, as well as an additional correlation through the path of expectations for those variables under learning. This paper is focuses on the latter phenomenon.

The interdependence between learning behaviour about beliefs about inflation and beliefs about output gap is coming from the general equilibrium linkages between these variables. To pin down the future evolution of inflation in this setting, it is not enough to understand the evolution of inflation expectation: it is also necessary to understand agents' expectations about output gap and vice versa. This observation is not surprising because the expected level of output gap contains additional information for the future path of expected and realized inflation. For example, for a firm to choose its reset price in (14) it must understand current and future demand for its products, which depends on the expected path of aggregate price level and the aggregate production in the economy. In an economy in which agents expect an inflation level of 2% and a small positive output gap, this would be different from an economy in which the same expected level of inflation is accompanied by beliefs about a large negative output gap. As a result, it is intuitive to anticipate that the paths of future expected and realized inflation are going to differ for those two cases.

Figure 4: Mappings from perceived to actual law of motion of output gap



To demonstrate how beliefs about inflation influence the path for actual output in the economy, the left panel on Figure 4 plots beliefs about output gap and its actual realizations along trajectories produced by O.D.E (40) given three different initial values for inflation expectations. In order to construct the left plot of Figure 4, I am using calibration of Bullard and Mitra (2002) and setting the Taylor rule coefficients to generally accepted levels: $\varphi_\pi = 1.5$ and $\varphi_z = 0.5$.[19] The horizontal axes represent expectations about an intercept for output gap a_z in (37) and the vertical axes shows the actual values of output gap, which realize given joint expectations about inflation and output gap along trajectories. The path of inflation expectations for each starting value also follows the trajectory derived from (37). Even though this path is not presented in the figure, it is used in order to construct the corresponding values of actual output gap.

Numbers on each panel on Figure 4 mark the location of realized output gap, given different levels of expected inflation. Note that the beliefs about output gap are initialized in the same way: agents' expectations are moved to a unitary level, marked by the dotted line on Figure 4. Each blue curve that starts from a number stands for a trajectory of T-map for beliefs about output gap and represents an evolution of actual output gap in time. The arrows define the direction of this dynamic given these initial conditions. The dots along each trajectory marks the location of beliefs for the first 20 periods after the economy starts evolving given each initial condition. It becomes apparent that the beliefs together with realized values of output gap move relatively fast in the direction of the 45-degree line, however once they have reached this area - the movement slows down significantly.

All curves, which represent trajectories of beliefs and realized values of output gap for a given initial condition, lead to the same destination − the REE −; however, the route along each is very different in terms of the actual path of output gap. Not only the starting values of the curves on Figure 4 are influenced by beliefs about the other variable, but the entire future path of beliefs and actual realisations of output gap are pinned down by *joint* movements of beliefs about inflation and output gap.

The first observation from the left panel on Figure 4 is that inflation expectations are crucial for the evolution of the real side of the economy. If inflation expectations are high enough (starting values 1 and 2), even given expectations of a positive output gap, the actual production level drops down today and continues to decrease for some time. The second observation is that the convergence of output gap back to the REE is non-monotonic in its expectations. Initially, for some starting value of beliefs (1 and 2), output gap overshoots in the opposite direction from equilibrium prior to returning to REE. The reasons for this overshoot will be studied in detail later in this section. Finally, it is no longer the case that output gap expectation matches (or even gets close to) the realized values only in the REE (REE neighbourghood).[20] It is clear that on the way to REE prior to changing the direction of movement,

[19]Refer to Section 2.1 for further details about calibration.

[20]at the same time, inflation expectations differ from actual inflation, because REE is still the only equilibrium for the economy under learning.

agents' expectations about output gap coincide with its realized value, and the exact location of this event depends on inflation expectations and might be far away from REE, as shown in Figure 4.

5.2 Mechanics of the model under learning

An economic intuition behind these dynamics is as follows. Suppose that in Period 1, expectations about output gap are in equilibrium; however, for an exogenous reason, inflation expectations happen to be higher than their equilibrium value by an amount Δ, $\Delta > 0$. Through a price-setting decision of firms in the economy, reflected in the Phillips curve (24), this leads to an upward pressure for inflation in Period 1, and the inflation rate increases by exactly $\delta\Delta$. At the same time, the central bank is setting a nominal interest rate according to the Taylor rule (25), and the described sequence of events dictates an increase in the nominal interest rate that is proportional to the increase in the inflation rate, $\varphi_\pi\delta\Delta$. The optimal output gap, which comes from an intertemporal consumption choice that is behind a dynamic IS equation (23), is defined not only by future expectations about output gap, which is assumed to be in equilibrium, but also by the real interest rate in this economy, which is given by the nominal interest rate net of inflation expectations.

It follows that the most important effect of the policy intervention that creates the real costs of inflation beliefs as unanchored in the REE is the effect on the real interest rate. An actual change in the real interest rate in Period 1 is given by $(\varphi_\pi\delta - 1)\Delta$, which can be either positive or negative, depending on the value of the Taylor rule coefficient on inflation. A policy that does not lead to a drift in the real rate requires $\varphi_\pi = \frac{1}{\delta}$, as was already explained above. However, if monetary policy is overreacting to expectations-driven fluctuations in the inflation rate, setting $\varphi_\pi > 1/\delta$ when $\Delta > 0$, this policy increases the real interest rate, which in turn implies a decrease in today's consumption/output gap. This applies to the US calibration and generally accepted value of the Taylor rule coefficient on inflation $\varphi = 1.5$.

It is also interesting to understand how the inflation-expectations-driven decrease in actual output gap accelerates and persists before returning to equilibrium. Given the adaptive nature of inflation expectations, it is impossible to bring them back to equilibrium within one period by manipulating the nominal interest rate. It follows that when the economy moves to Period 2, inflation expectations are still going to be significantly higher than in REE. However, in this period, output gap expectations are going to be driven downward by policy actions that have taken place in the previous period. In Period 1 the realized output gap was lower than expected, given that the actions of the central bank in that period intervened with intertemporal consumption-smoothing motives of rational economic agents. And rational agents, which are acting according to their perceptions (38)-(39), will decrease their expectations of output gap for the next period. So, in Period 2 the downward pressure on realized output is coming not only from a policy-initiated increase in the real rate, but also by lower-than-in-REE expectations about output gap (precautionary savings motive). The crucial part of the sequence of events presented above is that it takes only a couple of periods for agents' expectations about output gap to drop significantly. Because the agents are acting according to their expectations they are observing the decrease in actual output, which in turn makes them believe that their forecasts were reasonable.

In this setting, output expectations reinforce themselves in a downward spiral, and the actual output gap can be lower than in REE for several periods. However, when the output gap reaches its minimum[21] and the real interest rate becomes negative enough to outweigh the precautionary savings motive, the mechanism slowly unwinds:[22] agents begin to be more optimistic about future demand for their products, thereby correcting the path of actual consumption/output toward the equilibrium level. At the same time, the real rate returns to equilibrium. However, regaining the optimism about the future path of the output gap takes much more time than the initial decrease in output, because by construction this area of beliefs produces actual inflation and output gap as close to expected levels, even if inflation expectations are far away from REE. Therefore, agents do not become overly optimistic when output begins to increase on the way to equilibrium, implying that the inflation-expectations-driven loss in actual output might persist in this economy for a long time. If $\Delta < 0$ all the intuition is reversed. Consequently, output increases in the short run due to inflation expectations being lower than in REE.

5.3 Policy implications

As was previously explained, inflation expectations matter a great deal for the evolution of output gap in the economy, given the generally accepted values of Taylor rule coefficients. In this model economy, monetary policy is partially responsible for the extent to which inflation expectations influence the path of expected and actual output gap. This can be easily noticed from the equation for T-map (37) together with the expression for matrix B, each element of which depends on values of the Taylor rule coefficients.[23] An intuitive question is whether there exists a policy that can eliminate the interdependence of beliefs.

The policy that creates an evolution of output gap which is independent of a future path of inflation

[21] at this moment agents' expectations hit the eigenvector which corresponds to the largest eigenvalue

[22] because $\varphi_\pi > 1$ guarantees the expectational stability of this model economy

[23] Off-diagonal elements of matrix B are responsible for this interdependence.

requires setting the coefficient on inflation in the Taylor rule to an inverse of the discount factor. To make inflation an autonomous variable in this economy is more difficult, because this would require an infinitely strong reaction of nominal interest rate to deviations of output from the targeted level. This combination of policy recommendations sets off-diagonal elements of matrix B to zero and therefore makes evolution of each element of the dynamic system independent from one another. However, if we refer back to Figure 1 which demonstrates the value of the largest eigenvalue of B for each combination of policy coefficients, it is clear that this set of policy coefficients ($\varphi_\pi = 1/\delta, \varphi_z = \infty$) would never be preferable if the criteria for policy recommendations is the value of h_1.

The policy that eliminates dependence of output from current and future paths of inflation ($\varphi_\pi = 1/\delta$) expectations results in the largest eigenvalue of T-map being equal to the discount factor. These results are formally stated in Proposition 2:

Proposition 2.

$\forall \varphi_z$ if $\varphi_\pi = 1/\delta \Rightarrow \partial a_z/\partial a_\pi = 0$ and $h_1 = \delta$.

Proof of Proposition 2. Follows from $\partial a_z/\partial a_\pi = 1 - \delta\varphi_\pi$ and (44) below.

The discount factor is usually assumed to be close to one, and, therefore by construction, this policy leads to a very slow rate of convergence of beliefs about both variables back to equilibrium given any initial condition away from REE. The central panel on Figure 4 confirms that this policy makes trajectories for output gap independent of inflation expectations: what used to be distinct trajectories for every initial condition for inflation expectations in the case of baseline Taylor rule, now transforms into a single blue curve.[24] The second observation is that for the case of the policy suggestion stated in Proposition 2 beliefs about output gap move much faster to the neighbourhood of the REE compared to the baseline calibration. However, at the same time inflation is going to stay away from REE for longer, but as it is shown of the central panel on Figure 4 - this kind of expectations-driven inflation does not lead to any output costs.

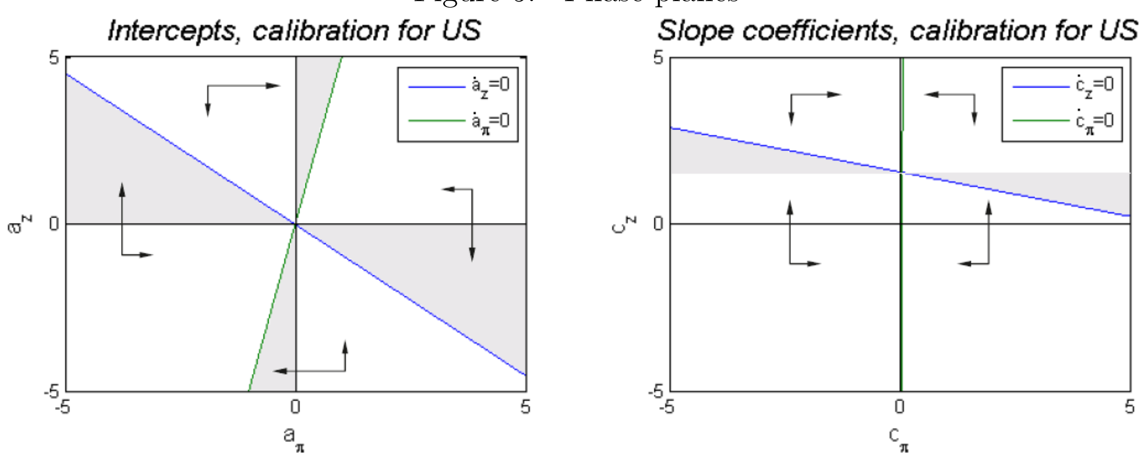
Beliefs about output gap are highly dependent upon the path of inflation expectations, and this interdependence in this simple model economy is controlled by monetary policy. It was already explained that this interdependence can be eliminated by applying a very particular policy recommendation stated in Proposition 2. What is left to be shown is whether the policy can exaggerate this relationship as well. To give the sense of how strong the policy effect is for interdependence of beliefs, I perform a comparative-

[24]Note that for this exercise I am only changing the value of φ_π compared to baseline calibration

statics exercise: I change Taylor rule coefficient for the output gap in the baseline calibration to a level that guarantees the lower value of h_1 in this economy $-$ higher rate of asymptotic convergence $-$, choosing φ_z from a blue area in Figure 2 given the baseline value of $\varphi_\pi = 1.5$. This policy would ensure that if beliefs are initialized away from their REE values they will return more rapidly to equilibrium. All other parameters, including the starting values for beliefs and the Taylor rule coefficient for inflation, are fixed at the same levels.

The right plot in Figure 4 shows the results of this exercise and demonstrates that this policy sends agents' beliefs even farther away from equilibrium when compared to a baseline Taylor rule. Now the same misperception of inflation expectations results in much higher real costs/gains both instantaneously and in terms of medium-run dynamics. Given any initial condition for beliefs, output gap now increasingly overshoots prior to returning to equilibrium. Therefore, by increasing the rate of convergence of beliefs to equilibrium, the policy at the same time makes output gap very sensitive to inflation expectations. Mainly, in this setting higher-than-in-REE inflation expectations result in higher real losses, which are a by-product of monetary policy, which is overly aggressive with regard to fighting deviations of inflation from the targeted level and at the same time ignores deviations of output gap from the targeted level.

Figure 5: Phase planes



To obtain a better understanding of this source of interdependence of learning behaviour, it is useful to look at phase diagrams for learning coefficients in the model economy. Figure 5 presents phase planes of learning coefficients for the case of baseline calibrations for the US. Compared to Figure 4, this allows to study dynamics of beliefs about both variables given any value of inflation expectations, however the realized values of output gap are no longer present on the Figure.

The left plot on Figure 5 sketches dynamics of constant terms in (29) and the right one describes the evolution of slope coefficients. Green and blue lines represent eigenlines for the corresponding dynamic systems: for the case of constant terms - eigenlines of B and for the case of slope coefficients - eigenlines of ρB. The eigenlines intersect at the REE values of learning coefficients, and, given the e-stability principle, the equilibrium is a stable node. The blue line shows the eigenline V_1 associated to the largest eigenvalue of matrix $B - h_1$. The arrows on each plot mark the direction of movement of beliefs if they are initialized in the corresponding area of the plane.

The eigenvectors presented in Figure 5 give an idea of to what extent learning about coefficients of inflation is influenced by acquiring knowledge regarding coefficients of output gap, and vice versa. In the ideal scenario, with an absence of interdependence, the eigenlines would be aligned with the axes $-$ the blue one with the horizontal axis and the green one with the vertical axis. That would guarantee that once out of equilibrium, all beliefs would move in the direction of equilibrium. The shaded regions in Figure 5 represent the areas from which beliefs about either inflation or output gap are sent in the wrong direction from equilibrium. This type of dynamic complicates learning and increases the amount of time in which agents' beliefs about some variables are not moving closer to REE and it generates different dynamics in the model due to learning.

The analytical expression for both eigenvalues and eigenvectors of matrix B, which describes the dynamic of intercepts in policy function (37), can be easily derived from (37). Their values are defined by the following equations:

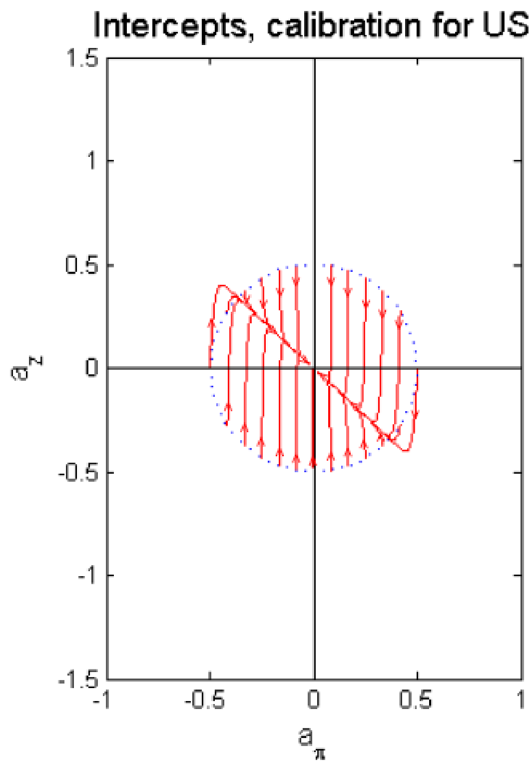
$$h_{1,2} = \frac{\sigma + \delta\sigma + \kappa + \delta\varphi_z \pm \sqrt{-4\delta\sigma(\sigma + \kappa\varphi_\pi + \varphi_z) + (\sigma + \delta\sigma + \kappa + \delta\varphi_z)^2}}{2(\sigma + \kappa\varphi_\pi + \varphi_z)} \tag{44}$$

$$v_{1,2} = \left[\frac{\sigma - \delta\sigma - \kappa - \delta\varphi_z \pm \sqrt{-4\delta\sigma(\sigma + \kappa\varphi_\pi + \varphi_z) + (\sigma + \delta\sigma + \kappa + \delta\varphi_z)^2}}{2\sigma\kappa}, 1 \right] \tag{45}$$

As it was already explained in the previous section, the eigenvector v_1 (45) associated to the largest eigenvalue h_1 (44), plays a very important role in this model economy. By definition its slope dictates the long-run inflation-output trade-off in the model. For the case of US calibration, from Figure 5 we can see that the eigenvector is close to a 45-degree line. This implies that long-run inflation expectations that are lower than in REE by $x\%$ result in $x\%$ proportional gains in output gap along the path of beliefs about both variables achieving equilibrium.

It is easy to verify that the corresponding eigenvector can be upward- or downward-sloping, depending on the interest rule coefficients. A policy that reacts to deviations of inflation from the targeted level just slightly more than one-to-one by changing the nominal interest rate can be beneficial if the inflation expectations happen to be higher-than-in-REE. If the policymakers set the Taylor rule coefficient on inflation, φ_π, to an inverse of discount factor, the eigenvector becomes aligned with the horizontal axes and as a result there are no real costs to inflation expectations being unanchored in the REE. If the policy manages to fix the Taylor rule coefficient on inflation in the following region $[1, \frac{1}{\delta}]$, the eigenvector becomes upward-sloping and therefore in the long run, higher-than-in-REE inflation expectations result in output gains.

Figure 6: Phase portraits



To understand the model dynamics in the short run it is necessary to take into account all eigenvalues and eigenvectors. For this purpose Figure 6 plots phase portraits of beliefs' dynamic for the same model economy. The starting values for beliefs are defined in a circle, with a unitary radius to give an idea of belief paths for any possible location of initial conditions. The dots along each trajectory marks the location of beliefs for the first 10 periods after the economy starts evolving given the initial condition. It becomes apparent that the economy moves very quickly to the blue line - eigenline V_1 associated to h_1. This is because in this model the difference between the two eigenvalues is so large that the influence of the smaller eigenvalue for the direction of movements of beliefs dies out within the first couple of periods. Therefore, not only in the long-run but also in the medium run, V_1 defines the output costs of higher-than-in-REE inflation beliefs for virtually any initial condition for beliefs.

6 Optimal Taylor rules

In the short run, model behaviour of the economy under learning differs substantially from the case when the same model is analyzed in the REE. Consequently, a set of Taylor rule coefficients that is optimal under learning might not coincide with the set that is considered to be so under REE. The current section verifies this statement and performs welfare analyses for the same model under both REE and learning.

An optimal Taylor rule is designed in a way that conducted policy maximizes welfare in the economy. A standard welfare criterion (negative of loss function) that is microfounded and is generally used for analyzing welfare in DSGE models is given by the following expression:

$$W = -E_0 \sum_{t=0}^{\infty} \delta^t (\pi_t^2 + \eta z_t^2) \tag{46}$$

Welfare losses are defined by the weighted sum of squared current and expected future deviations of inflation and output gap from their targeted levels. Coefficient η describes a relative weight that the central bank assigns to stabilization of output gap, compared to stabilization of inflation. If $\eta = 0$, monetary authorities care only about stabilization of inflation and ignore variations in output gap. Under rational expectations, the optimal Taylor rule would be a set of coefficients, φ_π and φ_z, that maximize W given all the equilibrium conditions (28). Under learning, the central bank will be solving the same problem; however, it will be restricted not only by the equilibrium dynamics of the economy−in this case given by (36)−but it also must simultaneously take into account initial conditions and the resulting evolution of beliefs under learning, while setting Taylor rule coefficients in (16).

An exact maximum for this problem depends on a precise calibration of an economy and might require an unfeasibly strong response of nominal interest rate to deviations of inflation and output gap from targeted values. It is more useful to define a set of policies that are preferable to others, given reasonable values of Taylor rule coefficients. Instead of deriving an analytical solution for this problem that is excessively complicated, I am assessing (46) in simulations, using a grid for Taylor rule coefficients.[25]

[25] $\varphi_z = 0.01 : 1/5 : 10$; and $\varphi_\pi = 0.445 : 2/25 : 5$;

An expression for the welfare criterion (46) can be decomposed into a weighted sum of net present values (NPV) of squared inflation and output gap. Therefore, for each point on a grid I am calculating an average negative NPV for each variable, according to the following expressions:

$$E(\pi^2) = -\frac{1}{nsim}\sum_{n=1}^{nsim}\left(\sum_{t=0}^{200}\delta^t(\pi_{t,n}^2)\right); E(z^2) = -\frac{1}{nsim}\sum_{n=1}^{nsim}\left(\sum_{t=0}^{200}\delta^t(z_{t,n}^2)\right) \tag{47}$$

Therefore, a preferable set of policy rule coefficients should be maximizing a weighted sum of negative NPV of squared inflation and output gap across simulations.

Figure 7: Contour of NPV for inflation and output gap under RE

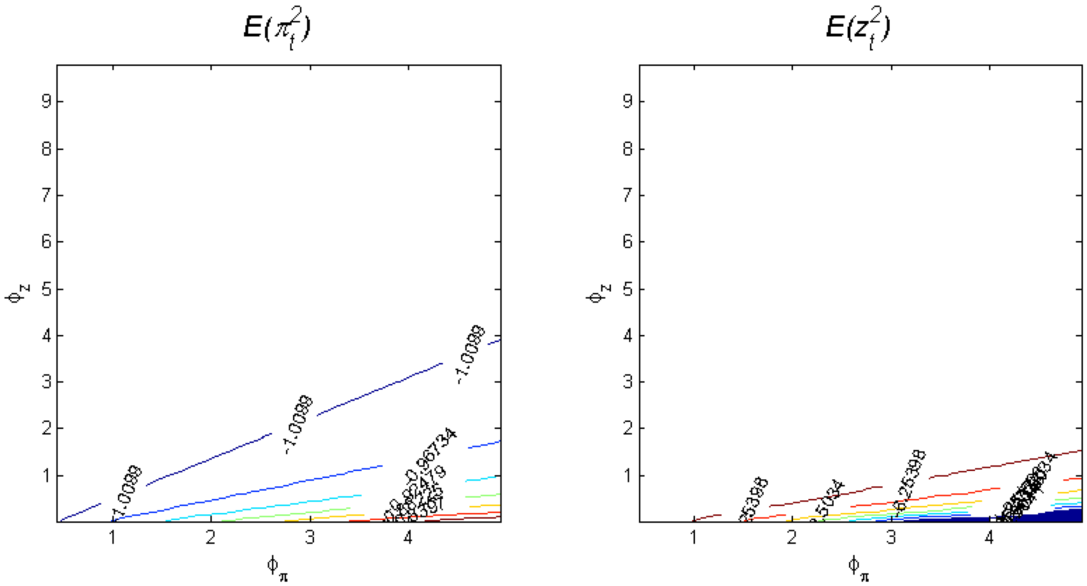


Figure 7 presents contours of (47) in the space of Taylor rule coefficients for the case when the model is studied in the REE. We are interested in the red region, which corresponds to higher levels of welfare or, symmetrically, lower losses for the economy, resulting from variability in either inflation (left panel in Figure 7) or output gap (right panel in Figure 7). Figure 7 confirms standard welfare results: if monetary policy is mainly concerned about variance of inflation, it should be reacting sufficiently by nominal interest rate to deviations of inflation from its target. The higher level of Taylor rule coefficients guarantee inflation stability and, therefore, lower welfare losses. Stabilization of output gap requires exactly the opposite actions from the central bank: some reaction to variations in output gap by the nominal interest rate is needed to guarantee its stability. However, only very high values of Taylor rule coefficients on inflation that are not supported by almost any reaction to output gap lead to high output losses under REE. This area is marked by dark blue region on the right panel in Figure 7.

Policy implication changes substantially if the same economy is studied under learning. If in the REE all variations in the economy were driven by exogenous disturbances, demand and cost push shocks; now inflation and output gap might also be separated from their targeted levels because of movements in agents' beliefs about them. In this setting monetary policy will be responsible not only for offsetting exogenous disturbances in the economy, but also for assuring the correct and fast movement of beliefs toward their equilibrium values.

Figure 8: Contours of NPV for inflation and output gap under learning, beliefs on trajectory

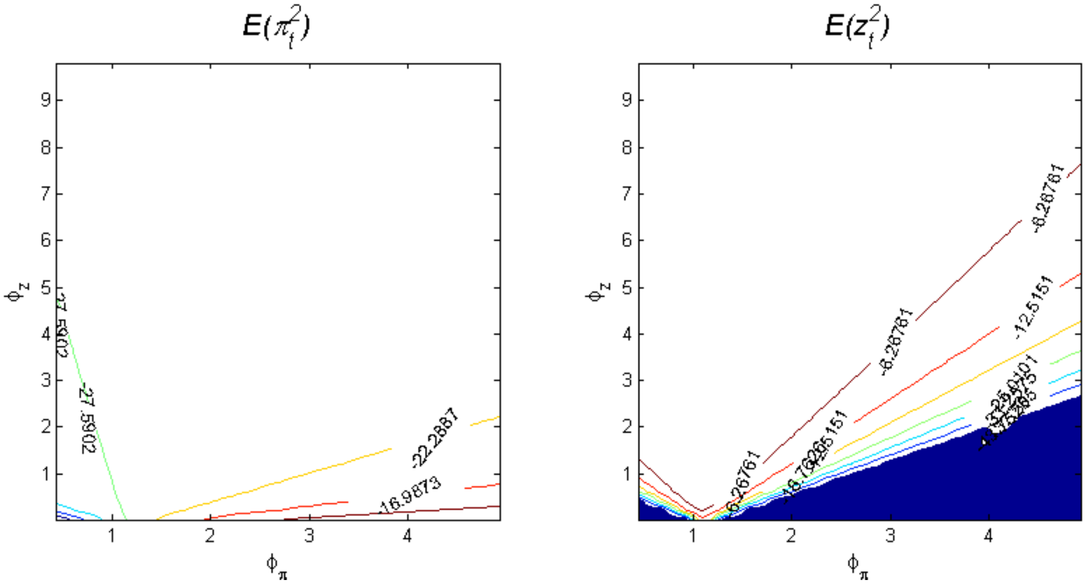


Figure 8 presents the results of welfare exercise if beliefs about inflation and output gap follow the trajectories, as described in Figure 9. Agents' expectations are shocked in a way in which both output gap and inflation are expected to be higher than in equilibrium. In this set of conditions welfare losses generated by agents' beliefs are significantly higher than those generated by exogenous shocks. This can be verified by comparing Figures 7 and 8: for every pair of Taylor rule coefficients the level of welfare is higher if the model is studied in the REE compared to the case when the same model is analyzed under learning, and beliefs are initialized away from their equilibrium values.

If monetary authorities are concerned only about inflation stability (left panel in Figure 8), policy suggestions correspond to the case of REE. Stronger response by nominal interest rate to deviations of inflation from its target will ensure that welfare is maximized in this economy. But under learning the same level of welfare (NPV of inflation) also allows for positive and quite large values of Taylor rule coefficient on output gap. In other words, some reaction to output gap does not instantaneously produce welfare losses from variations in inflation.

The right panel in Figure 8 presents welfare losses that are coming from variability in output gap. It turns out that if monetary authorities are even slightly interested in output gap being stabilized, the preferred area of Taylor rule coefficients differs substantially from the optimal set under REE. If agents are acting according to their expectations, in this model monetary policy is responsible for the path of beliefs about output gap when inflation expectations happen to be destabilized. And losses that are coming from beliefs about output gap overshooting prior to converging to their equilibrium value are much higher than losses associated with demand and cost push shocks in this setting.

The red area in Figure 8 corresponds to the set of policy rule coefficients for which the path of beliefs about output gap does not react excessively to inflation expectations. For lower values of Taylor rule coefficient on output gap, a reaction by interest rate to variations in inflation that is slightly stronger than one-to-one will assure that beliefs about output gap are not sent farther away from equilibrium in instances when they happen to be unachored. This result is not surprising because a value of Taylor rule coefficient on inflation that is equal to an inverse of the discount factor in this model guarantees that the path of beliefs about output gap are independent from inflation expectations. However, the same rule will make beliefs about inflation highly dependent upon the expectations of output gap. Therefore, an optimal policy depends on the relative weight that monetary authorities assign to stabilization of inflation and output gap. However, the reaction to inflation by a policy rate that is too strong can not be optimal for any preferences (values of $\eta > 0$ in (46)) that assign at least some positive weight to output gap stabilization.

7 Conclusion

This paper derives standard DSGE model under the assumption of internal rationality and demonstrates how model dynamics changes given the presence of learning in otherwise standard model. Mainly, this simple modification allows for persistent deviations of expected and actual inflation and output from their long run equilibrium.

Policy recommendations change substantially relative to rational expectations setting of the model. It follows that surprising agents with a higher level of inflation than expected results in a short-run increase in output gap. The extent to which output gap increases is also controlled by monetary policy and can be exploited by ignoring output expansion by policy rate. However, this strategy might be risky, because if agents expect inflation to be higher than its realised value−this will drive the economy into a recession, and the severity and persistence of the recession will depend on policy parameters. A suggested response should be that policy promptly reacts by aggressively stimulating output and temporarily allowing inflation to be higher than its target.

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A Proof of Proposition 1

Proposition 1

If limits for bond holdings are close to zero, labor and consumption choices are close to their REE values.

Proof

The optimal consumption/labor choice is given by the solution of the following problem

$$\max_{\{C_t,N_t,B_t\}_{t=0}^{\infty}} E_0^{\mathcal{P}^h} \sum_{t=0}^{\infty} \delta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{48}$$

subject to

$$C_t + \frac{B_t}{(1+r_t)P_t} - \frac{B_{t-1}}{P_t} \leq \frac{W_t}{P_t} N_t + \frac{D_t}{P_t} \tag{49}$$

$$\underline{B} \leq B_t \leq \overline{B} \tag{50}$$

taking r_t, P_t, B_t, W_t and D_t as given. Denote $\Delta B_t = \frac{B_t}{(1+r_t)P_t} - \frac{B_{t-1}}{P_t}$.

To prove Statement 1 it is necessary to follow the steps, outlined below:

1. Maximization of a concave function given a convex constraint leads to a unique solution $C^*(\Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and $N^*(\Delta B, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and, by implication, in optimum the budget constraint (49) binds. Therefore, the unique solution satisfies the following system of equations:

$$\frac{N_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \tag{51}$$

$$C_t + \Delta B = \frac{W_t}{P_t} N_t + \frac{D_t}{P_t} \tag{52}$$

2. Given that $r_t \gg -1$ and P_t is bounded away from zero, from the definition of ΔB it follows that as the bounds for bond holdings $[\underline{B}, \overline{B}]$ converge to zero, $\Delta B- > 0$.

Formally: $\underline{B} = -\epsilon$ and $\overline{B} = \epsilon \Rightarrow \lim_{\epsilon \to 0}(\Delta B) = 0$.

3. Solving (51) for labor:

$$N_t = \left[\frac{W_t}{P_t} C_t^{-\sigma}\right]^{\frac{1}{\phi}} \tag{53}$$

and substituting for the budget constraint allows us to define the following function:

$$f(C_t, \Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) \equiv C_t + \Delta B - \left(\frac{W_t}{P_t}\right)^{1+\frac{1}{\phi}} C_t^{\frac{-\sigma}{\phi}} + \frac{D_t}{P_t} \tag{54}$$

which in turn defines the optimal consumption choice:

$$C^*(\Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) : f(C_t^*, \Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) = 0 \tag{55}$$

4. By the Implicit Function Theorem

$$\frac{\partial C}{\partial(\Delta B)}|_{\Delta B=0} = -\frac{\partial f/\partial(\Delta B)}{\partial f/\partial C} = -\frac{1}{\partial f/\partial C} \tag{56}$$

Therefore, to finish the proof it is necessary to demonstrate that $\partial f/\partial C|_{\Delta B=0}$ exists and it is invertible.

Taking the derivative of (54) with respect to consumption, and evaluating at $\Delta B = 0$ leads to the following expression:

$$\partial f/\partial C|_{\Delta B=0} = 1 + \frac{\sigma}{\phi}\left(\frac{W_t}{P_t}\right)^{1+\frac{1}{\phi}} C_t^{\frac{-\sigma}{\phi}-1} > 0 \tag{57}$$

which is strictly positive given the non-negativity of consumption and a real wage.

Therefore, it follows that when $\Delta B \to 0$, the consumption choice converges to the REE, $C^* \to C^*(0, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and, from (53), it also follows that the labor choice converges to REE, $N^* \to N^*(0, \frac{W_t}{P_t}, \frac{D_t}{P_t})$.

Given that household understands the mapping of fundamentals into wages and dividends, the equilibrium consumption equals to aggregate output. This can be easily seen by substituting (21)-(22) into (54), which leads to $C_t^* = Z_t$.

B Non-stochastic steady state

In the steady state all real variables are constant, but nominal variables (P_t and X_t) grow at the same exponential rate $1 + \pi^*$. From (15) it follows that in the steady state:

$$MC_{t+k|t} = MC_{t|t}(1+\pi^*)^{\frac{\epsilon\alpha k}{1-\alpha}} \tag{58}$$

Therefore:

$$A = \widehat{A}; \tag{59}$$

$$1 + \widehat{r} = \frac{1+\pi^*}{\delta}; \tag{60}$$

$$\widehat{\left[\frac{P^*}{P}\right]_t} = \left[\frac{1-\theta(1+\pi^*)^{\epsilon-1}}{1-\theta}\right]^{\frac{1}{1-\epsilon}}; \tag{61}$$

$$\widehat{S} = \frac{(1-\theta)\widehat{\left[\frac{P^*}{P}\right]}^{\frac{-\epsilon}{1-\alpha}}}{1-\theta(1+\pi^*)^{\frac{\epsilon}{1-\alpha}}}; \tag{62}$$

$$\widehat{MC_{t|t}} = \frac{\epsilon-1}{\epsilon}\left[\frac{1-\theta\delta(1+\pi^*)^{\frac{\epsilon}{1-\alpha}}}{1-\theta\delta(1+\pi^*)^{\epsilon-1}}\right]\widehat{\left[\frac{P^*}{P}\right]_t}; \tag{63}$$

$$\widehat{Z}_t = \left[\left[(1-\alpha)\widehat{MC_{t|t}}\widehat{S}^{-\varphi}\right]^{1-\alpha}\widehat{A}^{1+\varphi}\widehat{\left[\frac{P^*}{P}\right]}^{\epsilon\alpha}\right]^{\frac{1}{\sigma+\varphi+\alpha(1-\sigma)}} \tag{64}$$

C Eigenvalue representation of a dynamic system

All standard tools of analyzing monetary policy rules described above are based on the eigenvalue representation of a dynamic system. If we are interested in asymptotic properties of the system the knowledge of eigenvalues is sufficient to define whether the equilibrium is a stable or unstable node. For my analysis of short run dynamics of agents' beliefs under learning it is crucial to keep track of the objects that are actually converging with the rate defined by eigenvalues.

If the eigenvalues are real, an eigenvalue representation of a system will define linear combinations of the original variables, which dynamics are independent from each other. For every starting point in the original coordinates, the direction of the movement of the system will be defined by the eigenvectors, and the scale of this movement will be defined by corresponding eigenvalues. If eigenvalues are real and one is significantly larger than another, the path of the system will be defined by the eigenvector that corresponds to the largest eigenvalue.

If the eigenvalues are complex, the movements in beliefs are going to be defined by a corresponding rotation-scaling matrix and a matrix that transforms the initial dynamic system to a rotation-scaling form. Technically, if eigenvalues of matrix B are complex, this matrix can be represented in the following form: $B = CAC^{-1}$, where A is a rotation-scaling matrix with a scaling parameter that is equal to an absolute value of the eigenvalues of B, and this parameter defines the asymptotical properties of the dynamic system. However, for the short run dynamics, both the rotation matrix and the transformation matrix C matter greatly. If norms of vectors produced by columns of T are significantly different, this matrix will be responsible for a large amplification of movements of the system in several directions.

D Dynamics of the economy in simulations

Williams (2014) proves that that for small values of the gain parameter, dynamic of beliefs follows an O.D.E. In this section I am going to check whether this result holds for the calibration which is used in this paper and a behaviour of beliefs, which is qualitatively similar to the one described in a previous section, is produced by an actual algorithm for sequential updating of beliefs, given by (38)-(39), in the presence of the full set of shocks.

To demonstrate that principle, I am simulating the economy of US 10,000 times. In Period 0, the beliefs of agents are initialized in a given point away from REE, but in the same period the actual rate of inflation and the output gap are still defined by their REE values. The constant terms are moved from rational expectation values, $[0,0]$, to $[0.1,0.1]$ and the slope coefficients are reallocated to $[c_z^{RE}+1, c_\pi^{RE}+1]$, where c_z^{RE} and c_π^{RE} are defined by (28). In other words, agents are expecting both inflation and output gap to be higher than their equilibrium values. They also expect a stronger reaction of both variables than in the REE to movements in natural rate of interest. Constant gain learning starts at Period 1, and beliefs are updated according to (38)-(39).[26]

Figure 9: Mean dynamics of coefficients in simulations, $nsim = 10000$

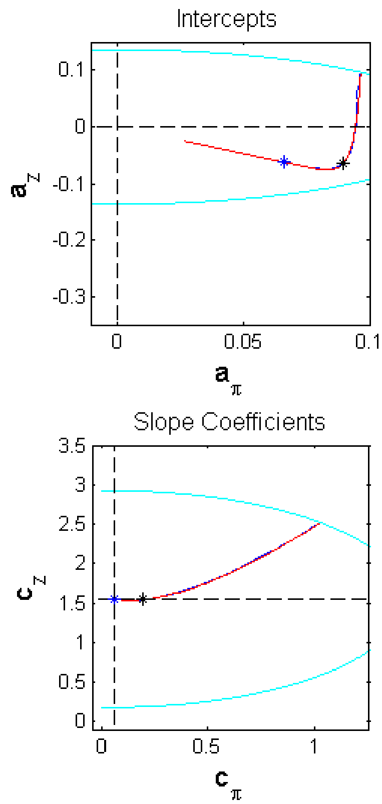


Figure 9 summarises the results of this procedure and plots the mean dynamics of learning coefficients.[27] The mean dynamics is presented by the blue curve, and the blue star on each plot marks the location of learning coefficients 100 years after the initial shock to beliefs. The red curve presents the trajectory of coefficients produced by a small O.D.E. (40), given the same initial values for beliefs. It follows that the mean dynamics of coefficients in simulations for the full version of the model almost perfectly follow the trajectories of beliefs as defined by the small O.D.E (40). Light blue circles on each plot represent the combination of values of learning coefficients for inflation and output gap, with the

[26]to initialize R in (38)-(39) I am using 10 years of simulated data

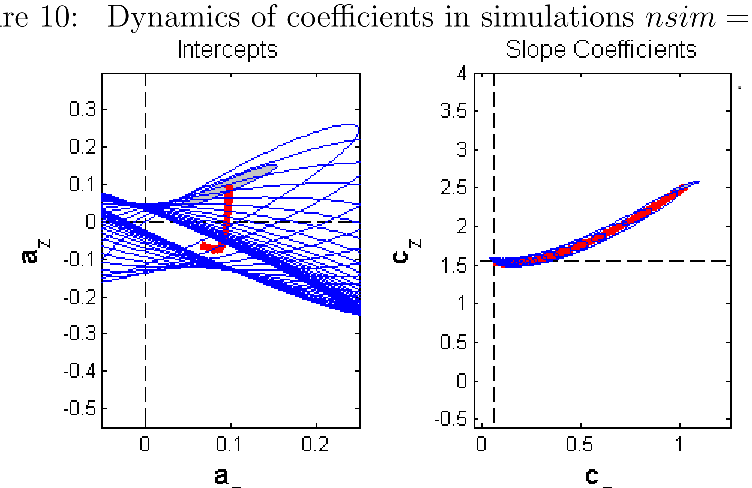
[27]Expected values of learning coefficients are approximated by their average values across independent simulations: $E_t(a_{\pi,t+1}) = \frac{1}{nsim}\sum_{i=1}^{nsim} a^i_{\pi,t+1}$

same distance to equilibrium as in the initial conditions for beliefs. So, the radiuses of the corresponding circles for both economies are equivalent.

Figure 9 confirms that trajectories produced by a small O.D.E. are a reasonable approximation of the behavior of beliefs in simulations. One of the implications of this result is that the mean dynamic of beliefs about constant terms are independent from the mean dynamic of beliefs about slope coefficients. In a given simulation, the dynamic of intercepts will be correlated with the dynamic of slope coefficients only through the path demand and cost push shocks. Therefore, it follows that to generate a significant drop in output, it is enough that inflation expectations are higher than the equilibrium value. This source is independent from the path of the natural rate of interest, as well as disturbances to the economy.

Another observation is that output reacts relatively quickly to destabilized inflation expectations. The black star on each graph marks the location of beliefs after only a quarter of the total number of periods had passed, and the blue star marks the location of beliefs after 400 periods had gone by. When beliefs about output gap were sent in the wrong direction from equilibrium, it is apparent that future correction in agents' perceptions takes a long time. This is an implication of the shape of the T-map that is presented in Figure 4: after some time the T-map becomes close to a 45-degree line, meaning that agents stop making large forecasting mistakes even if their beliefs are far away from REE. As soon as these beliefs are close to the actual realizations of variables, the learning slows down substantially; the resulting drop in beliefs about output gap and in actual output might persist for decades. And the source of this persistence is not a small value of constant gain parameter in (38)-(39) but rather a fact that the difference between realized and expected values of output gap diminishes very quickly. This prevents agents from discovering REE values of coefficients in their perceived law of motion in (29).

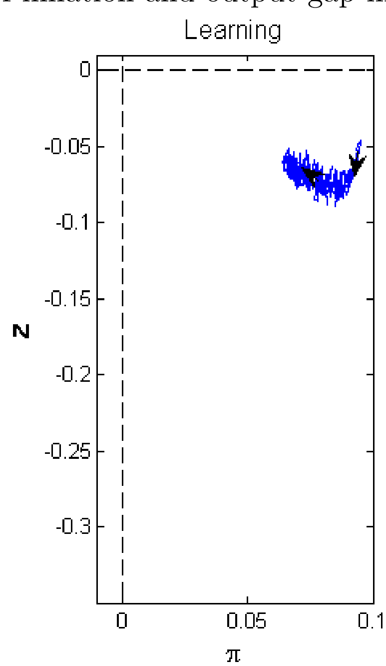
Figure 10: Dynamics of coefficients in simulations $nsim = 10000$



It is also informative for considering not only mean dynamics, but also an entire distribution of beliefs over time. In Figure 10 the 95% confidence bands are added to the mean dynamics path of learning coefficients. The shaded ellipse on Figure 10 represents the initial distribution of beliefs. The initial dispersion of beliefs is also produced by the calibration of each economy; I am assuming the same variances for initial noise in REE. The ellipse mainly depends on the autocovariance function of the exogenous process for a natural rate of interest, as well as variances of demand and cost-push shocks. With this exercise one can track how the initial dispersion of beliefs transforms under the presence of learning in the model.

Interestingly, if one solves the model under an assumption of rational expectations, an equilibrium dynamics of the economy dictates a small positive correlation between beliefs about inflation and output gap − that is confirmed by shapes of shaded ellipses in Figure 10. However, if agents actually take into account their expectations while making optimal decisions − that is, we consider the same model under learning − the correlation between beliefs about variables becomes negative. The reason behind this is that agents' forecasting mistakes are driven not only by exogenous disturbances, but also by trajectories for beliefs on the way to equilibrium. And these trajectories result in a strong negative correlation between beliefs about inflation and output gap, as was demonstrated in Figure 6. This dependence dominates the positive correlation that comes from exogenous shocks to the economy.

Figure 11: Mean path of inflation and output gap in simulations, nsim=10000



The results not only about the actual series for inflation and output gap from O.D.E dynamics are also confirmed in the simulations. Figure 11 presents the mean dynamic of actual inflation and output gap under learning if beliefs are incorrectly specified, in the same manner as discussed above. In the REE, the mean dynamic for both variables would be concentrated around zero, because unconditional expectations of inflation and output gap are zero, given that all disturbances, including the process for the natural rate of interest in the model are mean-zero. It follows that the dynamic that is presented on Figure 11 should be entirely produced by beliefs about constant terms in (29) being out of equilibrium, because all of the dynamics that are driven by the natural rate of interest are averaged out.

An actual output gap drops even faster than its beliefs in response to inflation expectations being higher than in RE. Given that it takes time for agents to realize that this drop is not related to other disturbances in the economy, their beliefs follow, with some lag, the dynamics of the actual variables. When beliefs are far away from realizations of actual variables, this lag is defined by the value of gain parameter in (38)-(39), which is a function of variances of shocks in the economy. However, once the output gap reaches its minimum value, agents' expectations catch up with actual dynamics. From now on the persistence of beliefs is dictated by agents' expectations being too close to actual realizations of variables, implying that the decrease in output persists even if the gain is calibrated to higher levels.

Figure 12: Mean path of deviations of inflation and output gap from REE in simulations, nsim=10000

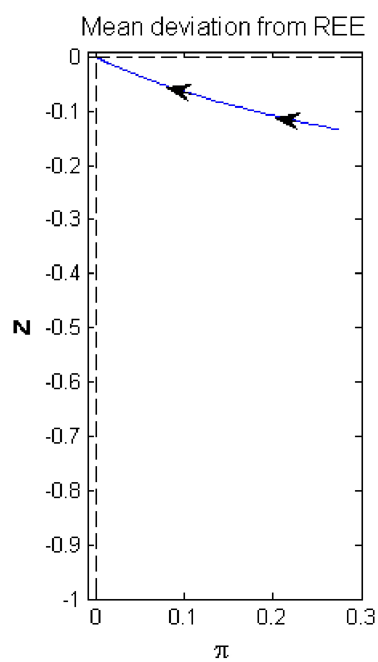


Figure 12 plots deviations of inflation and output gap from their REE paths that are driven by beliefs about slope coefficients that follow the trajectories described on Figure 9. To derive this path from series of inflation and output gap for each simulation, I am calculating the difference between dynamics of variables under learning and under REE for the same realizations of the shocks. Given that the resulting dynamics in actual series depend on realizations of the natural rate of interest, which are not restricted in any way in simulations, it makes it necessary to control for the sign of the natural rate while calculating deviations in inflation and output gap from their REE paths. In this manner I am also isolating these deviations from the path of beliefs about constant terms (29). Therefore, the results presented in Figure 12 correspond to positive realizations of natural rate of interest.

Figures 11 and 12 confirm that agents' beliefs that are out of equilibrium lead to significant deviations of realized inflation and output gap with respect to their paths in the REE. Therefore, studying this model under REE is uninformative, because these deviations−mainly a decline in output gap−are a result of a policy that is not considered to be problematic under an assumption of rational expectations.