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by Philippe Moutot

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ABSTRACT

In a stochastic pure endowment economy with money but no financial markets, two types of agents trade one non-durable good using two alternative types of cash constraints. Simulations of the corresponding variants are compared to Arrow-Debreu and Autarky equilibriums. First, this illustrates how financial innovation or financial regression, including systemic risk, may arise in a neo-classical model with rational expectations and may or may not be countered. Second, the price and money partition dynamics that the two variants generate absent any macroeconomic shock, exhibit jumps as well as fat-tails and vary depending on the discount rate.

Keywords: Financial development, Systemic Risk, Heterogeneity, Rational expectations, Monetary model, Cash constraints

JEL classification: E44

Non-Technical Summary

In this stochastic pure endowment economy with money but no financial markets, two types of agents trade one non-durable good under cash constraints. In a first variant, a cash-in-advance constraint "à la Clower" (1967) forces agents to own all the cash needed for settling upcoming purchases before the start of trading. In the second variant, a cash-at-the-end-of-the-day constraint allows agents to settle their transactions only once overall trading has ended. Simulations of these two variants are compared to the outcome of two more models. In the so-called Autarky model, each of our two agents lives in "autarky" i.e. transact in a separate centralized market with agents of his/her kind instead of sharing a centralized good market with the other type of agents. A shift from one of the Variants to this Autarky constitutes a *financial regression*. In the so-called Arrow-Debreu model, by contrast, markets are complete and centralized; i.e. agents can trade their endowments before the start of transactions. Moving from one of the Variants to the Arrow-Debreu model constitutes a *financial innovation.*

This illustrates how financial innovation or financial regression, including systemic risk, may arise in a neo-classical model with rational expectations and may or may not be countered. First, sufficient incentives and the corresponding legal apparatus are necessary to make the rational expectations equilibrium in Variant 1 sustainable. Moreover, whenever an unexpected change of the endowment's process makes the above-mentioned legal apparatus obsolete without prompting its adjustment in due time, a switch to the Autarky model may occur for certain partitions of money across agents. With Variant 2 by contrast, this is never the case. Secondly, financial innovation under Variant 1 may occur only when the distribution of money across agents is not too unequal while under Variant 2, the two agents always support financial innovation. Moreover, transitions from Variant 1 to Variant 2 and the reverse can happen only for one precise partition of money across agents to which convergence is in practice very rare. Furthermore, the best answer of authorities to systemic risk occurring in Variant 1 is not necessarily the adaptation of the legal apparatus that would make Variant 1 fully sustainable. On the contrary, it may be more astute for authorities who wish to foster financial innovation and overall welfare to promote a shift to Variant 2 in view of facilitating a subsequent shift to the Arrow-

Debreu model. Finally, if the original partition of money is too unequal when a single currency is proposed to two closed economies, the introduction of a such currency through cash in advance constraints does not provide a uniform improvement to economic agents in autarky.

Moreover, even absent any macroeconomic shock, the price and money partition dynamics that the two variants generate, exhibit jumps as well as fat-tails and vary depending on the discount rate. Consequently, as hoped by Farmer and Geanakoplos (2008) in their paper on "The virtues and vices of equilibrium and the future of financial economics'", it might be possible for such models, if complemented with further markets, agents and/or institutions to generate some of the dynamics that were initially pointed out by Cutler, Poterba and Summers (1987) or Schiller(1991). Such dynamics have often been associated to disequilibrium models advocated by so called "econo-physicists" such as Beinhocker (2006) and Bouchaud (2009).

Finally while tackling the above issues, a number of remarks confirming or infirming the literature on monetary models can be made. For example, contrary to Feenstra(1986) and Guidotti (1991), introducing a cash-in-advance constraint 'à la Clower' is not necessarily equivalent to including money in the utility function of agents. Also, the assumptions, made by other literature concerning the stability of wealth distributions do not apply in a models like ours. This explains why our results differ from others.

INTRODUCTION

The model presented in this paper is very simple. It includes two infinitely-lived agents with rational expectations living in a pure endowment economy, i.e. in an economy with neither production nor investment processes. Each period, agents receive their individual endowment of one non-durable good and then trade it in a centralized market. Also, this economy has neither financial markets nor financial intermediaries nor even an active central bank. However, there is a fixed quantity of money in the economy which agents, due to cash constraints, use both to carry out transactions and to save. As the endowments of the two agents differ and are determined by a Markov process, the model is dynamic and stochastic.

The model, the main characteristic of which is the interaction of heterogeneity with cash constraints, has two variants: in the first one, a cash-in-advance constraint "à la Clower" (1967) forces agents to own all the cash needed for settling upcoming purchases before the start of trading. In the second variant, a cash-at-the-end-of-the-day constraint allows agents to settle their transactions only once overall trading has ended[1].

However, despite or because of this extreme simplicity, the model exhibits properties that allow it to illustrate in a novel way the links between financial innovation and/or regression and systemic risk in the context of a monetary model. First, Variants 1 and 2 can be compared not only to each other, but also to two other general equilibrium models with close resemblance. In the so-called Autarky model, each of our two agents lives in "autarky'' i.e. transact in a separate centralized market with agents of his/her kind instead of sharing a centralized good market with the other type of agents. The shift from one of the Variants to this Autarky constitutes a *financial regression*. In the so-called Arrow-Debreu model, by contrast, markets are complete and centralized; i.e. agents can trade their endowments before the start of transactions. Moving from one of the Variants to the Arrow-Debreu model certainly constitutes a *financial innovation.*

Secondly, let us define systemic risk as the risk that an economy in equilibrium suddenly reaches a collectively less efficient equilibrium as a result of an endogenous

[1] Except for the possibility to interpret these variants as turnpike models, the model does not explain further how money arises in this economy and does not include mechanisms to explain the genesis of the corresponding legal and/or technical apparatus.

phenomenon possibly complementing or responding to an exogenous one. This definition is close to the one of Aglietta and Moutot (1993) and/or the Committee on the Global Financial System (2010). A shift away from a Variant to Autarky as a result of the decision of one agent following an exogenous shock therefore qualifies as a systemic event, although the model does not have banks or financial markets. Third, numerical solutions exist for both Variants. As shown in Moutot (1991), they can be found in almost all cases for Variant 1. Also, although the model is highly non-linear, the existence of solutions can be formally proven in specific cases. Variant 2, the cash-at–the–end-of the-day model, also has proven solutions in specific cases but in others does not have any. Moreover iterations do not converge when started in an inappropriate neighborhood. Nevertheless, for realistic values of the discount rate of utility, it is possible to numerically find and simulate equilibriums for Variant 2. As a consequence, numerical simulations allow calculating price and wealth distribution dynamics as well as agents' welfares[2]. Consequently, these properties allow considering *how financial innovation and/or regression may or may not arise in the context of a neo-classical model with rational expectations* and, in particular, *how the risk of a systemic event may arise and be countered.*

This is the main contribution of this paper. Indeed, as agents are fully rational, they can in principle make the same calculations as the author and the reader of this article. So agents may, depending on the state and the amount of money they own and barring further incentives of legal nature, be tempted to shift from one variant or model to another. For instance, agents may individually or collectively support the development of financial innovation as in the Arrow-Debreu model. Alternatively, they may prefer a shift to autarky, which barring obviating incentives, could occur if decided by one type of agents.[3]

As a consequence, the following results can be reported. First*, sufficient incentives and the corresponding legal apparatus are necessary to make the rational*

[2] In the example chosen, agents share a common logarithmic utility function and face an endowment process with micro- but no macro-economic uncertainty.
[3] It should be recognized however that agents are assumed to care only for their own consumption and do not share collective ideals They also assume that political or technical considerations and costs are not taken into account either. Neither are the computational difficulties that may be encountered in reaching a particular equilibrium considered. For instance, it may be argued that the costs associated to the identification of a fixed point solution to Variant 2 should be contrasted with those of Variant 1 and those of the Autarky and the Arrow-Debreu model which are negligible.
[5] For readers interested in this issue, Box 1 presents my views on this topic.

expectations equilibrium in Variant 1 sustainable. In Variant 1 moreover, *whenever an unexpected change of the endowment's Markov process makes the above-mentioned legal apparatus obsolete without prompting its adjustment in due time, a switch to the Autarky model may occur for certain partitions of money across agents. With Variant 2 by contrast, this is never the case.*

Secondly, the partition of money among agents in the economy is an essential element when making Pareto comparisons of the equilibriums generated by our models. In particular, *financial innovation under Variant 1 may occur only when the distribution of money across agents is not too unequal while under Variant 2, the two agents always support financial innovation*. *Moreover, transitions from Variant 1 to Variant 2and the reverse can happen only for one precise partition of money across agents to which convergence is in practice very rare.* Furthermore, *the best answer of authorities to systemic risk occurring in Variant 1 is not necessarily the adaptation of the legal apparatus that would make Variant 1 fully sustainable.* On the contrary, it may be *more astute for authorities who wish to foster financial innovation and overall welfare to promote a shift to Variant 2 in view of facilitating a subsequent shift to the Arrow-Debreu model.* Consequently, the framework used for the assessment of systemic risk in this suite of simple models is useful in assessing trade-offs between the prevention of systemic risks and financial development.

Finally*, if the original partition of money is too unequal when a single currency is proposed to two closed economies, the introduction of a such currency through cash in advance constraints does not provide a uniform improvement to economic agents in autarky*, which is an important qualification of the Townsend (1980) and Kiyotaki and Wright (1989) results. Again, this is not the case with cash-at-the-end-of-the-day constraints.

The following should be recognized however. While allowing to generate systemic events in connection with the partition of money/wealth and while providing an estimate of its long term cost for each agent*, this framework does not allow a description of all aspects of such systemic event. For instance, it does not tell us how long this event would last nor the behavior of prices and consumption during this interim period.* This is because it does not make hypotheses concerning the functioning of the economy while moving from one type of equilibrium to another, i.e it does not model disequilibrium. This is also at odds with the more concrete approaches of systemic risk followed by authors like Acharya (2009) when developing a theory of systemic risk.

Nevertheless –and it is a second but less important contribution of this paper- the two variants of this general equilibrium model are able to generate diverse dynamics of money/wealth partition across agents and of prices. These dynamics exhibits jumps and fat-tails, depending on the level of the discount rate of utility. Consequently, as hoped by Farmer and Geanakoplos (2008) in their paper on "The virtues and vices of equilibrium and the future of financial economics", it might be possible for such models, if complemented with further markets, agents and/or institutions to generate some of the dynamics that were initially pointed out by Cutler, Poterba and Summers (1987) or Schiller (1991). Such dynamics have often been associated to disequilibrium models advocated by so called "econo-physicists" such as Beinhocker (2006) and Bouchaud (2009).

Proving or illustrating these various points however assumes first that one accepts the use of cash constraints in a model as legitimate although a number of economists have questioned such practice[5]. Moreover, it necessitates a good understanding of the way both Variants operate and, in particular, of the dynamics of money partition across agents as well as of the dynamics of prices under the two Variants. Somehow, those dynamics of money/wealth partition condition not only the shifts to Autarky but also the practicability of collective decisions in favor of financial development. Furthermore, describing and explaining these dynamics is also an occasion to confirm or infirm a set of points and/or assumptions previously made in the literature on money demand and/or on cash-in advance models as well as former literature on models with heterogeneous agents and the ability of general equilibrium models to generate price dynamics with realistic features.

For example, according to Feenstra (1986) and Guidotti (1991), introducing a cash-in-advance constraint 'à la Clower' in a model would be equivalent to including money in the utility function of agents. We show however that whenever agents are sufficiently heterogeneous and for realistic values of the discount factor of utility of agents, cash constraints cannot remain continuously binding, making this equivalence uncertain. Hence money may be held even when cash constraints are not binding and may therefore play a role of asset or insurance on top of its role in transactions.

This point is particularly important for macro-modeling as most micro-economically founded macro-models either introduce money within the utility function of economic agents or combine the use of a cash-in-advance constraint with the assumption of a representative agent, thereby making the cash-in-advance constraint continuously binding and implying a constant money velocity. Even when introducing state-of-the art investment functions, and although constraints may occasionally be non-binding, this makes velocity insufficiently flexible as demonstrated in Hodrick, Kocherlakota, and Lucas (1991) in the case of a representative agent's model. Therefore some (e.g. Woodford 2006) reject this inclusion and argue that money plays no active role in the determination of the economic dynamics. By contrast, the most frequently used DSGE models with an active role for money (for instance Christiano, Motto and Rostagno 2007) not only include money in the utility function of consumers but complement it with the inclusion of banks and investors for which continuously binding collateral constraints are imposed. In the case of Variant 1 however, the absence of complete financial markets and the frictions generated by the cash-in-advance constraint make money a natural saving and insurance instrument and makes money velocity very flexible.

This is in line with views already put forward by Bewley (1980) and Lucas (1980) and the subsequent literature with heterogeneous agents. Indeed, our approach rather supports similar views by Hansen and Imrohoroglu (1990) and Fuerst (1991), although the latter also mentions a preference to avoid such role for money. By contrast, other more recent approaches by Algan and Ragot (2010) and by Wen (2010) also give a role of insurance to money. Also, the importance of timing conventions apparent in the contrasts we present between Variant 1 and 2 was pointed out by many of the authors above, starting with Lucas (1980). For instance, we find a shift from Variant 1 to Variant 2 clearly diminishes the short term variability of prices while increasing their average level and thereby decreasing velocity.

However, why do all those papers not mention results similar to ours in terms of price dynamics or systemic risk? This is because our modeling strategy differs from these various approaches in several respects. First, we consider two distinct agents instead of using the methodology originally suggested by Lucas(1980) which lumps into one family the various agents but as a consequence cannot consider the impact of wealth on agents' behavior. Moreover, we do not introduce money in the utility function like Algan and Ragot as counterparty to also assuming continuously binding cash constraints. Also, and although we introduce cash-in advance constraints and heterogeneity like Wen (2010), we do not have a bond market and a monetary policy. Moreover, contrary to all this literature on heterogeneous agents including Wen (2010), we do not assume that the distribution of money or wealth has to converge to a constant after some finite time. This assumption, which is made by many authors like Hansen and Imrohoroglu (1992) or Burkhard and Mauβner in their 2005 book on general equilibrium models is understandable with an infinity of agents and the possibility for monetary policy or other macro-economic policies to intervene in order to influence or stabilize such money/wealth distribution. This assumption, which is natural and fair with one representative agent as argued by Lucas (1980) and proven for fairly general statistical settings by Micio (2004), is a much less justified starting point in the context of financial development or systemic risk which by nature cannot be consistent with a stationary or even a constant distribution of money or wealth.

As a consequence, our two variants generate a high variability of the partition of money/wealth and of prices in cases where other models would not assume and/or generate any. In particular, microeconomic shocks invisible at a macroeconomic level may well create sizable variations of prices and money/wealth distribution. Moreover, they generate various jumps and fat-tail effects that are otherwise difficult to generate.

In the following, Section 2 presents the model and its two variants, shows how to identify the functional operator of which the fixed point determines completely the solution. Section 3 assesses whether and when cash constraints are binding or not. Section 4 describes the dynamics of the model by concentrating on a specific example and shows that they are quite diverse. Section 5 examines the link between cash constraints and welfare in the context of the example identified in Section 4. In particular, Section 5 shows the need for a legal framework to make the rational expectations equilibrium sustainable, and illustrates the existence of systemic risk, and financial development in the context of the very simplified suite of models described above. Section 6 concludes.

Section 2

A MODEL WITH TWO AGENTS UNDER CASH CONSTRAINTS,

ONE GOOD, ONE CURRENCY AND NO BOND MARKET

In this section, I describe the two variants of the model, define its equilibrium, outline the corresponding first order conditions and transform them into a functional operator. I also discuss the solutions of this operator and the numerical techniques used to simulate the model.

Description of the model

The model is in its first version a generalization of two well known models. Although it is similar to the one-agent monetary model with cash in advance constraint developed by Lucas and Stokey (1987) or Coleman (1986), it has two agents and can be interpreted as a turnpike model along the lines first developed by Townsend (1980). However, instead of being a perfect foresight model like Townsend's and Manuelli and Sargent's (1988), it incorporates uncertainty using a stochastic framework borrowed from Lucas and Stokey (1987). When it was formulated in my own thesis work (Moutot 1991), it was also a forerunner of models like Hansen and Imrohoroglu (1992) which include heterogeneous agents and cash constraints. However, as explained in its summary offered by Burkhard and Mauβner (2004), most of this heterogeneous agents' literature makes the assumption that the distribution of money across agents is constant or at least continuous over some range as soon as the economy has converged toward a stationary equilibrium. While this assumption is natural in the representative agent case or if some policy enforces it, it is more tentative with heterogeneous agents as explained already by Lucas (1980). However, it allows identifying equilibriums and calculating them numerically without first solving for a fixed point of a functional operator, as shown below. This is not assumed here.

The model is formulated in discrete time with an infinite horizon. There are two agents, respectively named a and b and not an infinity as in Hansen and Imrohoroglu (1992). At the beginning of each period, t, each of these two agents receives an endowment of a unique non-storable good. These endowments called respectively ξ_t^a and ξ_t^b are outcomes of a stochastic process (to be defined later) such that ξ_t^a and ξ_t^b are bounded away from zero. The sum of ξ_t^a and ξ_t^b, which is the economy total endowment of goods for period t is called ξ_t.

There is no private or asymmetric information. At the beginning of each period each consumer observes his own endowment as well as the other consumer's endowment. Hence the information set I_t of the two consumers is identical and contains data on past and present endowments and prices. Each agent has preferences over his/her infinite lifetime consumption sequence $\{c_t^i\}_{t=0}^{\infty}$ as described by its time-separable utility function,

$$E\left[\sum_{t=0}^{\infty} \beta^t U_i\left(c_t^i\right) \middle| I_t\right] \quad i \in \{a,b\}$$

where $0 < \beta \leq 1$ is identical for agent a and b, but where $U_a(.)$ and $U_b(.)$ can differ. Both $U_a(.)$ and $U_b(.)$ are assumed to be continuously differentiable, strictly increasing and strictly concave.

The only asset in this economy is money. The total amount of money in the economy is fixed to 1 and, at any given time t, this amount is divided between the two agents. Agent a possesses m_t^a units and agent b possesses m_t^b units such that:

$$m_t^a + m_t^b = 1$$

From one period to the next, changes in the money holdings of the two agents are described by their budget constraints.

$$m_{t+1}^i = m_t^i + p_t\ (\xi_t^a - c_t^i\ ) \quad i \in \{a,b\}$$

Indeed, after receiving their endowments, the two agents go to a market. In the **first variant of the model**, they sell their entire endowment and buy the amounts of the good (respectively, c_t^a and c_t^b) that they want to consume during period t independently and at a competitively determined price p_t. Both need to own enough cash at the beginning of the day to finance their consumption independently of the prospective receipts of their endowment's sale. Therefore the possession of money at the beginning of period t is essential in this variant, for both consumers are confronted by a Clower-type (1967) cash in advance constraint: This is representative of an atomised market with no clearing authority, no possibility to divide and sequence purchase and sale orders, where trust is limited and legal guarantees on the payment of intra-day debts non-existent.

$$p_t \;\; c_t^i \leq m_t^i \quad i \in \{a,b\}$$

In the **second variant of the model**, they can simultaneously sell their endowment and buy at a competitively determined price p_t their consumption during period t and can use the proceeds of their sale to guarantee their purchases. The possession of money is therefore constraining only at the end of period t trading when they need to settle all their transactions, which implies

$$m_{t+1}^i \geq 0 \quad i \in \{a,b\}$$

or equivalently

$$p_t \;\; c_t^i \leq m_t^i + p_t \xi_t^i \quad i \in \{a,b\}$$

This, by contrast, implies that a clearing system and the legal and computational framework necessary to ensure its good functioning are available, although the reasons for its creation and the costs it generates are not accounted for by the model. The equations corresponding to this variant will be numbered with a ‘ sign whenever different from those of the first variant.

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Finally, the equilibrium of the good market requires that, at each period t,

$$c_t^a + c_t^b = \xi_t^a + \xi_t^b = \xi_t$$

Uncertainty is introduced through the definition of endowments. These endowments are time invariant functions of shocks generated by a first-order Markov process with a stationary transition function of density $\pi(.,.)$ such that

$\xi_t^i = \xi^i(s_t) \qquad i \in \{a,b\}$ and

$$P(s_{t+\mu+1} \in B | s_{t+\mu} = s) = P(s_{t+1} \in B | s_t = s) = \int_B \pi(s, ds')$$

whenever B belongs to the family of Borel Sets of S.

Definition of an equilibrium

Equilibrium in this economy is a set of processes $\{c_t^a\}$, $\{c_t^b\}$, $\{m_t^a\}$, $\{m_t^b\}$ and $\{p_t\}$ such that at any time $t, c_t^a, c_t^b, m_{t+1}^a, m_{t+1}^b$ and p_t be the solutions of the two following maximization problems supplemented by two general equilibrium conditions:

Problem of Agent $\quad i \qquad i \in \{a,b\}$

$$\underset{c_t^i, m_{t+1}^i}{Max} \quad E[\sum_{l=0}^{l=\infty} \beta^l \, U_i(c_{t+l}) | \, I_t] \qquad i \in \{a,b\}$$

subject to:

$$I_t = \{ (p_{t-u}, \xi_{t-u}^a, \xi_{t-u}^b) | u \in (0,1,2,...,t)\}$$

$$p_t \, c_t^i \le m_t^i \qquad i \in \{a,b\} \qquad (1.1)$$

$$p_t \; c_t^i \le m_t^i + p_t \, \xi_t^i \qquad i \in \{a,b\} \qquad (1.1')$$

$$m_{t+1}^i = m_t^i + p_t \, (\xi_t^i - c_t^i) \qquad i \in \{a,b\} \qquad (1.2)$$

General Equilibrium conditions

$$c_t^a + c_t^b = \xi \qquad (1.3)$$

$$1 = m_t^a + m_t^b \qquad (1.4)$$

Equilibria studied here are such that processes generated are time-homogeneous functions of shocks $\{s_t\}$ and of $\{m_t^a\}$ and do not depend on sunspot variables[6].

First order conditions

If we call γ_t^i and θ_t^i the four Lagrange multipliers corresponding to (1.1) or (1.1') and (1.2), we have the following first order conditions:

$$U_i'(c_t^i) - (\gamma_t^i + \theta_t^i)p_t = 0 \qquad i \in \{a,b\} \qquad (1.5)$$

$$\theta_t^i - \beta\, E_t(\theta_{t+1}^i + \gamma_{t+1}^i) = 0 \qquad i \in \{a,b\} \qquad (1.6)$$

Second order conditions

$$\frac{d\theta^i(s,m_t^i)}{dm_t^i} \le 0 \quad \text{and} \quad U_i''(.) \le 0 \qquad i \in \{a,b\} \qquad (1.5') \text{ and } (1.6')$$

Transforming the set of first-order conditions into a functional operator.

a) The determination of the current price p_t

If our model possesses an equilibrium, then p_t can be shown to be a function h(.) of m_t, the current partition of money between agents , of θ_t^a and θ_t^b, the Lagrange multipliers for the two budget constraints, and of $\dot{\xi}_t$, the vector of goods endowments.

To see this, let us first write (1.1) and (1.1') as respectively:

$$c_t^i \le m_t^i / p_t \qquad i \in \{a,b\} \quad \text{with equality when } \gamma_t^i > 0$$

[6] For further explanation, see Moutot(1991).
[8] For more detail, see again Moutot (1991).

$$c_t^i \leq m_t^i / p_t + \xi_t^i \qquad i \in \{a,b\} \text{ with equality when } \gamma_t^i > 0$$

Since $U_i(.)$ is strictly increasing and strictly concave, its derivative $U'_i(.)$ has an inverse $U'^{-1}_i(.)$. It follows from (1.5) that:

$$c_t^i = U'^{-1}_i(p_t(\theta_t^i + \gamma_t^i)) \qquad i \in \{a,b\}$$

Like $U'_i(.)$, U'^{-1}_i is strictly decreasing. Also, the Lagrange multiplier γ_t^i can only be positive or equal to zero.

Therefore,

$$c_t^i \leq U'^{-1}_i(p_t\ \theta_t^i) \qquad i \in \{a,b\} \text{ with equality when } \gamma_t^i = 0$$

Consequently,

$$c_t^i = Min[m_t^i / p_t, U_i'^{-1}(\theta_t^i\ p_t)] \quad i \in \{a,b\} \qquad (1.7)$$

$$c_t^i = Min[m_t^i / p_t + \xi_t^i, U_i'^{-1}(\theta_t^i\ p_t)] \quad i \in \{a,b\} \qquad (1.7')$$

Finally, substituting (1.7) or respectively (1.7') for $i = a$ and $i = b$ into (1.3), writing m_t^a as m_t and using (1.4) to replace m_b^t by $1 - m_t$, and taking into account that in the second variant both cash constraints cannot be simultaneously binding, we get:

$$\xi_t = Min\{\frac{1}{p_t}, \frac{m_t}{p_t} + U'^{-1}_b(\theta_t^b p_t), \frac{1-m_t}{p_t} + U'^{-1}_a(\theta_t^a p_t), U'^{-1}_a(\theta_t^a p_t) + U'^{-1}_b(\theta_t^b p_t)\}$$

(1.8)

$$\xi_t = Min\{\frac{m_t}{p_t} + \xi_t^a + U'^{-1}_b(\theta_t^b p_t), \frac{1-m_t}{p_t} + \xi_t^b + U'^{-1}_a(\theta_t^a p_t), U'^{-1}_a(\theta_t^a p_t) + U'^{-1}_b(\theta_t^b p_t)\}$$

(1.8')

ξ_t is therefore equal to the minimum of four different functions of p_t in the first variant and only three in the case of the second variant, each of them decreasing and invertible. Therefore there exists for both variants, a function h such that:

$$p_t = h(m_t, \dot{\theta}_t, \dot{\xi}) \qquad (1.9)$$

where $\dot{\xi}_t$ is the vector (ξ_t^a, ξ_t^b) and $\dot{\theta}_t$ is the vector (θ_t^a, θ_t^b).

The nature of this function $h(.)$ is best understood by inverting each of the four or three functions of p_t on the right-hand side of (1.8) or (1.8') independently and writing $h(.)$, as the minimum of these four, respectively 3, inverted functions.

$$h(m_t, \dot{\theta}_t, \dot{\xi}_t) = Min\{h_1(\xi_t), h_2(m_t, \theta_t^b, \xi_t), h_3(m_t, \theta_t^a, \xi_t), h_4(\theta_t^a, \theta_t^b, \xi_t)\} \qquad (1.10)$$

$$h(m_t, \dot{\theta}_t, \dot{\xi}_t) = Min\{ h_2(m_t, \theta_t^b, \xi_t^b), h_3(m_t, \theta_t^a, \xi_t^a), h_4(\theta_t^a, \theta_t^b, \xi_t)\} \qquad (1.10')$$

This shows that the price p_t shifts between 4, respectively 3, different regimes of determination, depending on which agent is or is not cash-constrained. $h_1(\xi_t)$ is the price which prevails when both consumers are under a binding cash constraint in the first variant and it is equal to $1/\xi_t$. $h_2(m_t^i, \theta_t^b, \xi_t)$, and respectively $h_2(m_t^i, \theta_t^b, \xi_t^b)$, defines p_t when consumer a's cash constraint is binding whereas consumer b's cash constraint is not. $h_3(m_t, \theta_t^a, \xi_t)$, respectively $h_3(m_t, \theta_t^a, \xi_t^a)$, determines p_t when agent b is under a binding cash constraint while agent a is not. Finally, m_t does not enter the function $h_4(\theta_t^a, \theta_t^b, \xi_t)$ because $h_4(.)$ defines the price p_t in the case where neither a nor b are bound by cash constraints.

b) The partition m_{t+1} of the money supply

Using (1.7), respectively (1.7') and (1.2), one can write:

$$m_{t+1}^i = m_t^i + p_t \xi_t^i - p_t\, Min\{\frac{m_t^i}{P}, U'^{-1}_i(\theta_t^i p_t)\} = Max\{p_t \xi_t^i, m_t^i + p_t(\xi_t^i - U'^{-1}_i(\theta_t^i p_t))\} \quad i \in \{a,b\}$$

(1.11)

$$m_{t+1}^i = m_t^i + p_t \xi_t^i - p_t\, Min\{\frac{m_t^i}{P} + \xi_t^i, U'^{-1}_i(\theta_t^i p_t)\} = Max\{0, m_t^i + p_t(\xi_t^i - U'^{-1}_i(\theta_t^i p_t))\} \quad i \in \{a,b\}$$

(1.11')

Substituting (1.9) into (1.11), respectively (1.11') shows that m_{t+1} is determined by the same variables as p_t and can consequently be written:

$$m_{t+1} = M(m_t, \dot{\theta}_t, \dot{\xi}_t) \qquad (1.12)$$

Functions $h(.)$ and $M(.)$ specify p_t and m_{t+1} as functions of m_t, θ_t^a, θ_t^b, ξ_t^a and ξ_t^b. To characterize the equilibrium law of motion m_{t+1} and hence p_{t+1}, we must solve for the Lagrange multipliers θ_t^a and θ_t^b.

Constructing the functional operator

We now make use of equations (1.6) and (1.11) for $i = a$ and $i = b$. Since both equations link variables at time t to expectations of variables at $t+1$, they will become the core of a functional operator which is outlined the assumption of "stationary expectations"[8]:

For this, we will simultaneously take into account the set of all possible draws of the stochastic process $\{s_t\}$ and of all possible partitions of money, therefore replacing s_t and m_t by two generic variables respectively called s and m. Then,

if $\quad s_t = s_{t+u} = s$ and $m_t = m_{t+u} = m$,

$\theta_t^i = \theta_{t+u}^i = \theta^i(s,m)$ for $t \geq 0$, $u \geq 0$, and $i \in \{a,b\}$

Let us also define $\theta(s,m)$ as:

$$\theta(s,m) = \begin{bmatrix} \theta^a(s,m) \\ \theta^b(s,m) \end{bmatrix}$$

and assume $\theta(s,m)$ belongs to C^2 x C^2 where C^2 is the set of continuous and bounded functions on S x [0,1].

From (1.5) and (1.7), respectively (1.7'):

$$\gamma_t^i + \theta_t^i = Max\{\theta_t^i, \frac{1}{p_t} U_i^{;}(\frac{m_t^i}{p_t})\}. \qquad i \in \{a,b\}$$

respectively
$$\gamma_t^i + \theta_t^i = Max\{\theta_t^i, \frac{1}{p_t} U_i^{;}(\frac{m_t^i}{p_t} + \xi_t^i)\}. \qquad i \in \{a,b\}$$

It follows from (1.6) and (1.12) that:

$$\theta(s,m) = \Phi(\theta)(s,m) \qquad (1.13)$$

where:

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'} U_a'(\frac{m'}{h'})\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'} U_b'(\frac{1-m'}{h'})\} \quad \pi(s,ds') \end{bmatrix}$$

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'} U_a'(\frac{m'}{h'} + \xi^a(s'))\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'} U_b'(\frac{1-m'}{h'} + \xi^b(s'))\} \quad \pi(s,ds') \end{bmatrix}$$

respectively,

with: $m' = M(m, \dot{\theta}(s,m), \dot{\xi}(s))$ and: $h' = h(m', \dot{\theta}(s',m'), \dot{\xi}(s'))$

Stationary equilibria can therefore be considered as the fixed points of a multi-dimensional functional operators Φ defined by (1.13). The knowledge of this fixed point combined to the knowledge of the stochastic process $\{s_t\}$ and of the initial partition of money m_0 determines all the other variables. Please note that at no time was the assumption of a constant distribution of money across agents made.

c) Existence and numerical calculation of solutions

Finally, do functional operators like Φ have solutions and can we calculate them?

The answer to the first part of this question is still incomplete. Indeed, the Schauder Theorem which is the standard tool used by mathematicians to prove the existence of fixed points cannot be directly applied here given hat Φ is not a compact operator. However, as shown in Moutot (1991), it is possible to prove in a number of Variant 1 cases that its solutions are also solutions of more convoluted but nevertheless compact operators derived from Φ and to which the Schauder Theorem can be applied. Hence, solutions to (1.13) can be proved to exist in specific cases, including when endowments are constant and when β is small enough. However, such convoluted operators have not yet been derived for the most general cases, implying that full certainty about the existence of solutions to (1.13) cannot be reached yet.

However, solutions can in most Variant 1 and Variant 2 cases be approximated by numerical techniques. Let us represent the range [0,1] over which m may vary by a grid with a finite number of points (usually 256 in the forthcoming sections) and the functions $\theta(s,m)$ as matrices of dimension (2,S,256) where S is the finite number of possible shocks. As also shown by Moutot (1991), it is possible to search for solutions by iterations of Φ occasionally combined with interpolations when it is not possible due to computing constraints to use a fine enough grid[10]. It is then shown that solutions can be found across a number of choices for ß, for $U_a(.)$ and $U_b(.)$, and for Markov processes such as $\xi^i(s_t) \quad i \in \{a,b\}$. This answers positively the second part of the question above and motivates the next sections.

[10] As $\theta(s,m)$ is a multidimensional function approximated by a matrix, such interpolations are based on a generalization of the traditional Newton technique to an infinite dimensional context. Suppose that $\theta_1(s,m)$ and $\theta_2(s,m)$ are two functions between which the iterations of the functional operator end up alternating. Be $\theta_3(s,m)$ such that:

$\theta_a(s,m) = a\ \theta_1(s,m) + (1\text{-}a)\ \theta_2(s,m) \quad 0 <\_a\_< 1$

and $\theta_3(s,m) = \underset{a}{Arg\min}\ (\theta_a(s,m) - \Phi(\theta_a(s,m))) * (\theta_1(s,m) - \theta_2(s,m))$

where * is the scalar product of in the space of finite dimension matrices used to approximate the functions to which applies. $\theta_3(s,m)$ and $\theta_4(s,m) = \Phi(\theta_a(s,m))$)can be determined by successive approximations. In a one dimensional case, this method is identical to the Newton interpolation method. In a multidimensional case however, this is only the first step of a sequential interpolation process, the same interpolation technique being applied to $\theta_3(s,m)$ and $\theta_4(s,m)$ in a second step. After enough iterations, this interpolation technique has in most cases led us to a point where $\theta_n(s,m)$ and $\theta_{n+1}(s,m)$ were close enough to be considered as good approximations of the fixed point.

[13] The relative level of the general welfare created by Variants 1 and 2 was checked in order to ascertain that differences between them did not result from the randomness associated to the Monte Carlo method. The relative levels were found to be estimated in a robust way.

Section 3

WHEN IS MONEY IN THE UTILITY FUNCTION EQUIVALENT TO A CASH CONSTRAINT?

One of the original reasons for developing the first Variant of the model presented in this paper is to answer the above question. Following Feenstra (1986) or Guidotti (1991), a cash-in-advance constraint is equivalent to the inclusion of money into the utility function. Indeed, if it is continuously binding, it is equivalent to maximize the utility function under such constraint or to add to the utility function a separable part including money. At the same time, continuously binding cash-in-advance constraints imply that the velocity of money in the economy remains constant over time, which is not realistic and explains the rejection of cash-in-advance constraints by many economists.

It is therefore important to understand whether and when cash-in-advance constraints are continuously binding. In this section, I will show, as demonstrated by Moutot (1991), that whenever the discount rate of utility is higher than a certain threshold, value, the two cash constraints of Variant 1 cannot be simultaneously and continuously binding. In Variant 2, both cash constraints cannot be simultaneously binding by construction. However, I will also show that, at least for a specific set of stochastic environments and when utility functions are logarithmic, solutions with one of the cash constraints always binding exist for β within a certain range. This will allow determining for which parameters the specific models used in Sections 4 and 5 have continuously binding cash constraints.

<u>Theorem 1</u>

Be $U_a(.)$ and $U_b(.)$ two continuously differentiable, strictly increasing and strictly concave utility functions. Assume that the model considered is specified as a Variant 1 model and that β is strictly superior to 0. If there exists a “stationary expectations" equilibrium and if, in this context, after a finite number of periods both agents are always under cash constraint, then:

$$\beta \le \beta^1_{Max} = \underset{\substack{s \subset S \\ s'' \subset S \\ i \in \{a,b\}}}{Min} \frac{\xi(s) U_i^{'} \frac{\xi^i(s'')}{\xi(s'')} \xi(s)}{\int_S \xi(s') U_i^{'}(\xi^i(s) \frac{\xi(s')}{\xi(s)}) \pi(s, ds')} \qquad \textbf{(2.1)}$$

Proof: see in Annex 1 copied from the full theorem proved by Moutot (1991) showing existence of solutions in such case.

It is easy to illustrate this formula in the case where both utility functions are logarithmic, i.e $U_a(.)$ and $U_b(.)$ are equal to log(.). (2.1) is transformed into:

$$\beta \le \beta^1_{Max} = \underset{\substack{s \subset S \\ s'' \subset S \\ i \in \{a,b\}}}{Min} \frac{\frac{\xi^i(s'')}{\xi(s'')}}{\frac{\xi^i(s)}{\xi(s)}} \qquad \textbf{(2.2)}$$

Hence, in Variant 1 and for logarithmic utility, the equivalence between money in the utility function and cash-in-advance constraints is only valid if the ratio of individual shares of the total endowment does not vary too strongly across time. If one assumes as found by most studies that the discount rate of utility is around 0.95 for quarterly models of the economy, this implies that the ratio of such shares across time should remain higher that 0.95. Equivalently, this share should never vary more than 5.2% across time. Obviously, this implies that the above mentioned equivalence is probably not valid continuously and likely not in financial crises where the partition of endowments may strongly vary.

In the context of Variant 2, cash constraints can by definition never be binding simultaneously as evidenced by the fact that $h_1(\xi_t)$ does not intervene in (1.10').

$$h(m_t, \dot{\theta}_t, \dot{\xi}_t) = Min\{ h_2(m_t, \theta^b_t, \xi^b_t), h_3(m_t, \theta^a_t, \xi^a_t), h_4(\theta^a_t, \theta^b_t, \xi_t)\} \qquad (1.10')$$

However, under Variant 1, cash constraints are binding whenever m is low or high enough whatever the state, because without money, consumption is impossible. Is there by analogy a level of money partition under Variant 2 under which each agent gets systematically constrained? The answer is no. Under Variant 2, one is not necessarily cash-constrained whenever he/she owns no money.

Theorem 2

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$ Any solution to (1.13') for which prices are well defined is such that:

-at least for one s in S, prices are determined by h_4 or h_3 for m=0.

-at least for one s in S, prices are determined by h_4 or h_2 for m=1.

.

Proof: See Annex 1.The intuition behind this result is the following: there must be a state where the endowment of agent a is higher than his/her usual or average consumption. Therefore, he/she will consume less than his/her endowment even if he/she has no money, and thereby will be in a position to acquire some money to be used in the next period. Consequently, m=0 is not necessarily a sign of cash constraint and, by definition of h_4 and h_3, this implies that one of them must determine prices in this case.

Finally, under Variant 2, it is interesting to check for which parameters there is at each point in time one cash-constrained agent. This determines "a contrario" for which values of β none of the agents is ever cash-constrained. Although this domain can be generally determined, (see Proposition A.3 in Annex 1), it does not take the shape of a simple and general formula like (2.1) or even (2.2). valid under most circumstances. This is why I chose to calculate and present it for the specific type of Markov process and type of endowment variability for which a simple formula is available. Theorem 3 below also offers an actual proof of existence of solutions to (1.13) for cases where there always is one agent at a time under cash constraint.

Theorem 3

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$. Suppose finally that the endowment matrix $\xi(.)$ is such that $\dfrac{\xi^a(1)}{\xi^b(1)} = \dfrac{\xi^b(2)}{\xi^a(2)}$=z.

Then a Variant 2 solution to (1.13) exists if $\dfrac{2z}{1+z} \leq \beta < \beta^2_{Max} = \dfrac{-2z + \sqrt{20z^2} + 16z)}{2(1+z)}$

and is such that cash constraints are always binding for one of the two agents.

Proof : See Annex 1.

Overall, limiting ourselves to the specific case of logarithmic utility, of two states with the above probability matrix and of endowments such that $\frac{\xi^a(1)}{\xi^b(1)} = z$, it is possible to summarize in Chart 1 below the findings on the binding character of cash constraints in Variant 1 and 2 solutions to (1.13) for alternative values of β and z.

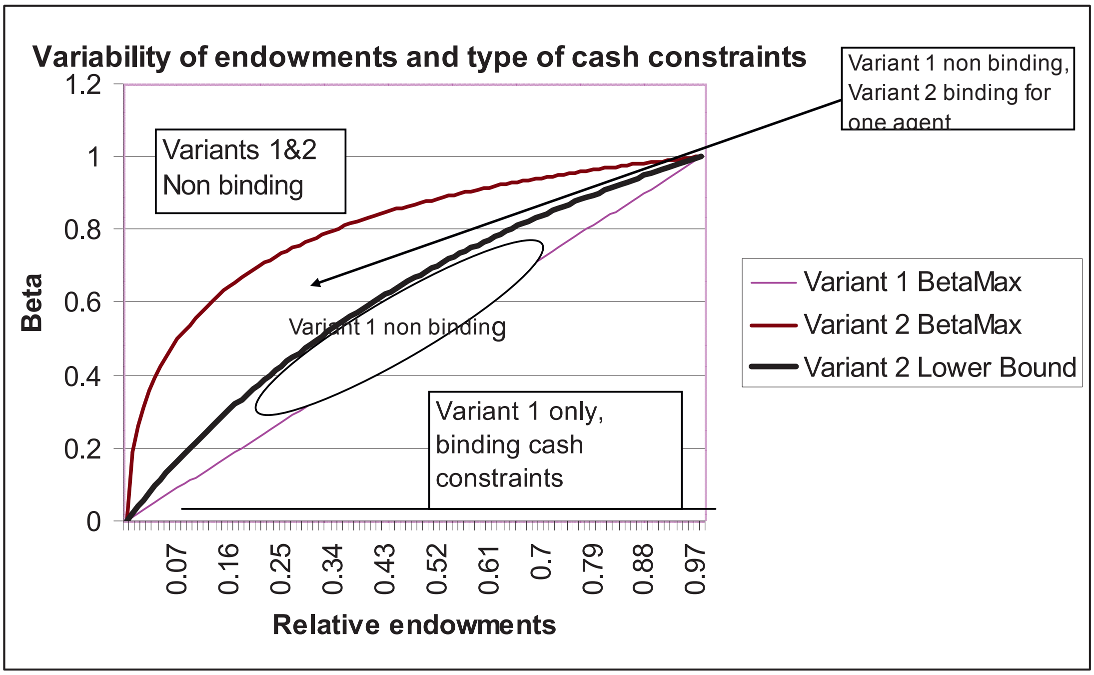


Chart 1

In the *area below the diagonal*, Variant 1 has solutions with binding cash constraints and defined prices. Indeed, (2.2) can now be written as: $\beta \leq \beta^1_{Max} = z$. By contrast, Variant 2 has no solution with determined prices. This does not mean that non-monetary equilibriums do not exist. They do but do not give a well defined value to goods and hence to money. Of course, the assumptions made by Feenstra (1986) and Guidotti (1989) cannot be relevant in such a case.

Above the diagonal, Variant 1 has solutions but they all include occasions when none of the cash constraints are binding. By contrast, above the diagonal, Variant 2 solutions with well defined prices exist only above the black curve. Up to the red curve, they always have one agent under cash constraint while the other is not constrained. Beyond, the red curve, they also include occasional episodes where none of the agents is cash constrained.

For instance, assuming β=0.95 again, z has to be lower than $\beta / 2-\beta$ or 0.90 and higher than 0.76 in order to allow for the existence of a monetary equilibrium with one binding constraint all the time. Consequently, for a given agent, his/her share of the total endowment $\frac{\xi^{a}(s)}{\xi\ (s)}$ varies between $\frac{\xi^{a}(1)}{\xi\ (1)}=\frac{1}{\frac{\xi(1)}{\xi^{a}(1)}}=\frac{1}{1+\frac{1}{z}}=\frac{z}{1+z}$ and $\frac{\xi^{a}(2)}{\xi\ (2)}=\frac{1}{\frac{\xi(2)}{\xi^{a}(2)}}=\frac{1}{1+z}$. Hence, its share of the total endowment may vary up to 31% from one state to another and still be consistent with the existence of one binding cash constraint but should vary more than 10% in order to be consistent with the existence of a monetary equilibrium. While some large variations are possible but limited ones impossible, such an environment is again not realistic and comes in contrast with the views of Feenstra(1986) and Guidotti(1991).

Hence, for realistic values of the discount factor and for logarithmic utility functions, *the equivalence between cash-in-advance models and models with money-in-the-utility function becomes uncertain* because, contrary to assumptions made by Feenstra (1986) and Guidotti (1989), cash constraints cannot remain continuously binding when the discount rate of utility is too high relative to economic uncertainty. Of course, this result is partly the consequence of an absence of banks or of a bond market in the models. However, even if the latter existed, they would not allow the two agents to invest all their excess cash in bonds all the time or to see their purchases fully financed unless the model would include an assumption of perfect foresight or would open and close the banks before and after each trading session. Hence, the cash constraints would not be continuously binding and this equivalence would remain uncertain.

Section 4

THE DYNAMICS OF PRICES AND MONEY PARTITION
UNDER VARIANTS 1 AND 2

How are the dynamics of prices and money partition affected by the heterogeneity of agents and the existence of non-continuously binding cash constraints? In order to illustrate their impact, I devise a simple experiment that allows for a comparison of the two variants of our model, one with the other as well as with other models. This lets us differentiate the impact of the limited financial constraint as reflected by Variant 2 from the impact of stronger financial frictions coming from cash-in-advance constraints, i.e in Variant 1. In this endeavor, I will first describe price variability in a general manner. Then I will examine the dynamics of money partition and explain how it is related to the dynamics of prices. I will then discuss the dynamics of money partition and in turn of prices. This will help show that such dynamics are much richer than in models with one representative agent and share some, but not all of the characteristics of price dynamics observed in reality. Some of these elements will be useful for the rest of the paper.

<u>The simulation</u>

Let us simulate the following case:

-both agents have a logarithmic utility function,

-both discount rates of utility are equal to 0.9,

-the stochastic environment is described by the probability matrix $\begin{bmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{bmatrix}$,

-the endowment is equal to: $\begin{bmatrix} 5-xs & 5+xs \\ 5+xs & 5-xs \end{bmatrix}$ with xs=2.

Then the total goods endowment in the economy is constant over time and fixed at 10 units per period. However, agents' individual endowments vary: as xs=2, each agent has one chance out of two to receive 3 units of good while the other receives 7 and one chance out of two to receive 7 units while the other receives 3 only. Hence ß is superior to β^1_{Max} , as:

$$0.9 = \beta \succ \beta^1_{Max} = \underset{\substack{s \subset S \\ s'' \subset S \\ i \in \{a,b\}}}{Min} \frac{\xi^i(s'') \big/ \xi(s'')}{\xi^i(s) \big/ \xi(s)} = 3/7,$$

and long term solutions cannot have constantly binding cash constraints in Variant 1.

Moreover, such value of β is also in the range of values for which equilibriums with well defined prices in Variant 2 have to include situations where none of the agents is cash constrained as shown by Theorem 3:

$$\beta^2_{Max} = \frac{-2z + \sqrt{20z^2 + 16z)}}{2(1+z)} .=0.835782 < 0.9$$

After having identified the value of the fixed points of

$$\theta(s,m) = \Phi(\theta)(s,m) \qquad (1.13)$$

we use it to calculate the values of $M(m, \dot{\theta}(s,m), \dot{\xi}(s))$ and: $h(m, \dot{\theta}(s,m), \dot{\xi}(s))$ for all possible couples (s,m) in S X [0,1]. Then we calculate time series by giving an initial value to m and iterating up to 1000 and occasionally 2000 periods.

$$m_{t+1} = M(m, \dot{\theta}(s_t, m_t), \dot{\xi}(s_t)) = F(m_t, s_t) \qquad (4.1)$$

<u>Money velocity in Variants 1 and 2 in relation to usual models</u>

If the total goods endowment of an economy and its total money supply are constant, traditional macro-economically defined money demand equations can only generate constant prices and constant money velocity. Similarly, in the case of a one representative agent model such as the one developed in Lucas and Stokey (1987) the steadiness of the total goods endowment and the absence of any monetary surprise also imply constant consumption, always-binding cash constraints for the representative agent, as well as constant prices and money velocity. However, such is not the case in our two variants if the partition of the total endowment between agents is not constant over time and if cash constraints are not always binding (see Figures 1 and 2 below).

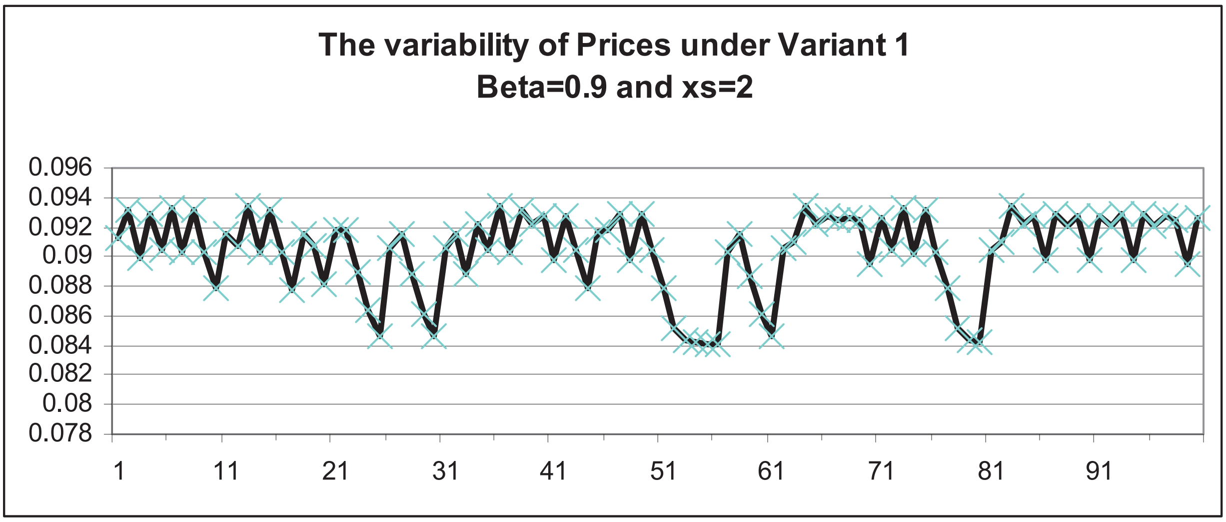


Figure 1

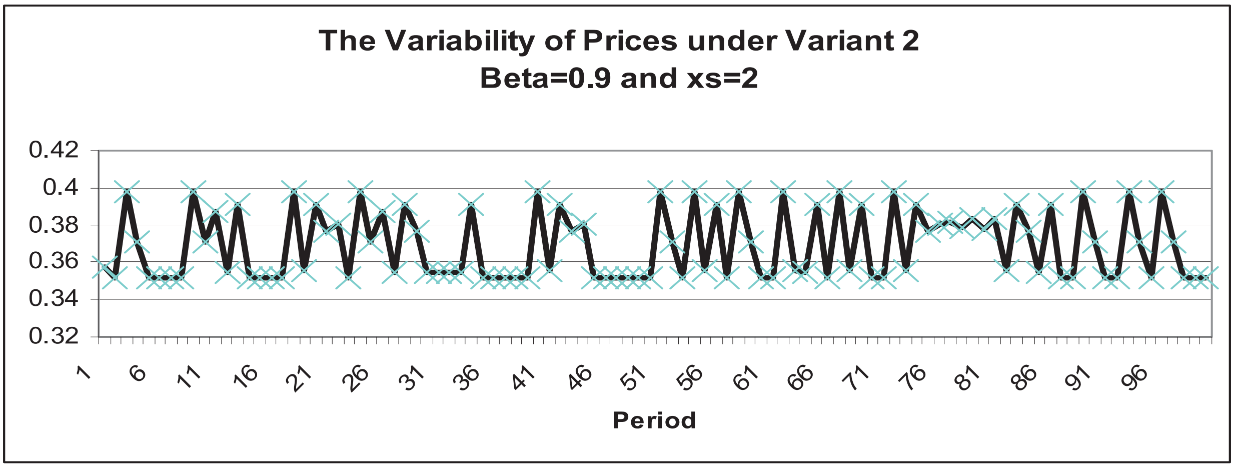


Figure 2

Indeed, the money partition, the price and therefore money velocity are highly variable in both Variants as evidenced in Figures 1 and 2. However, the level of prices is very different in the two variants. As apparent in Table 1, it is 4.7 times higher in Variant 2 which is logical as the price is connected to the utility of net transactions rather than to their bulk. Also, the average deviation, the standard deviation and, more importantly the variance of prices is much lower in Variant 1.

Finally, prices are more (and negatively) auto-correlated at the first lag in Variant 1. This negative correlation is logical but also shows the limitations of a model that, despite heterogeneity and financial frictions, cannot adequately mimic a widely-discussed feature of asset markets (see Cutler, Poterba, and Summers (1989) or Schiller (1991)), their positive autocorrelation in the short term..

Table 1

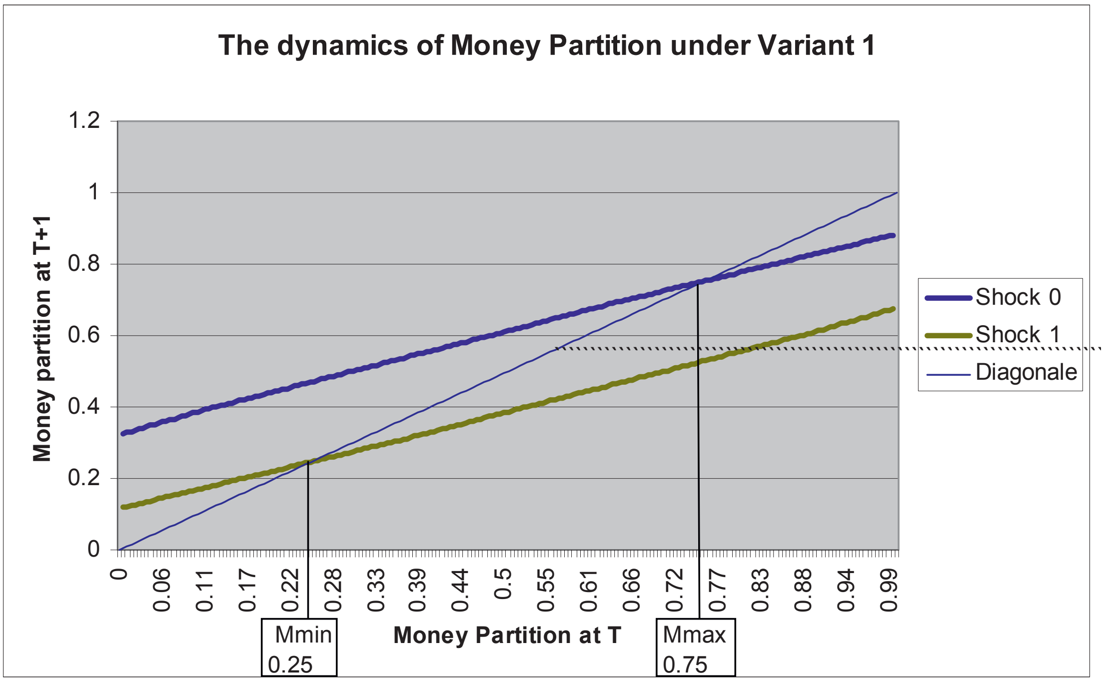
Prices	Variant 1	Variant 2	Ratio
Average	0.085653397	0.403005117	4.705069
Average deviation	0.005251789	0.030751014	5.85534
Standard deviation	0.006110022	0.033598821	5.498969
Variance	3.73E-05	1.13E-03	30.23866
Autocorrelation			
lag 1	-0.378095537	0.072960957	
lag 2	-0.070243947	-0.134459375	
lag 3	-0.090740589	-0.03038133	

Indeed, it includes neither investment nor loans and does not assume a positive correlation of its shocks. It therefore cannot offer its agents the possibility to envisage a continuation of recent trends in prices and therefore a positive short-term correlation. However, as we will see later, the model has some potential for exhibiting some other features of market data.

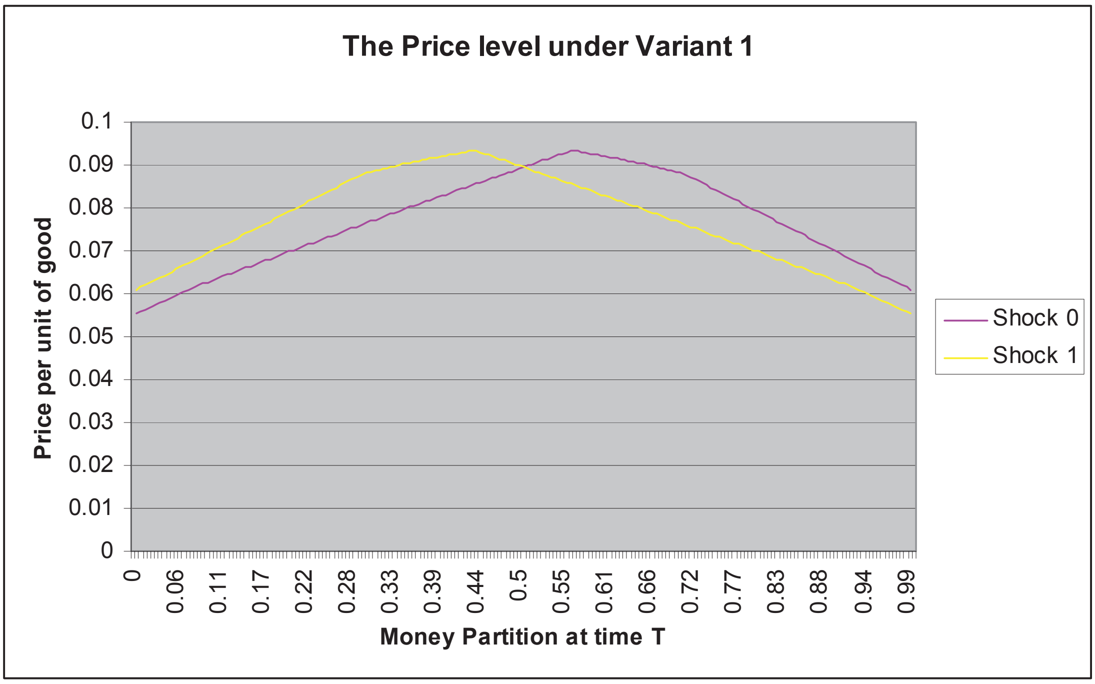
Explaining the dynamics of money partition and of prices in Variants 1 and 2

Indeed, the dynamics of money partition can be quite different across variants and evolve with the value of β . This is easy to see on Graphs 1.1 and 1.2 where the money partition of period t+1 is shown as a function of the money partition at time t and of the state of the world s at time t. In Variant 1 the money partition always converges (see also Graph 3.1) into an interval formed by the intersection of the diagonal with the two curves describing the evolution of money partition from one period to the next depending on the state of the world. Afterwards, it remains within this interval, i.e. between m_{Min} and m_{Max}, and may a priori reach any point of it depending on the succession of exogenous states s_t. Somehow, money partition jumps according to the state inside a box delineated by the interval $[m_{Min}, m_{Max}]$ and two almost parallel and increasing lines.

This is quite different under Variant 2 (see Graph 1.2 as well as Graph 3.2). Whatever β, the partition of money may always become so unequal that it concentrates into the hands of one agent and reach 0 or 1. Moreover, the distribution of money partition across time may show accumulation at 0 and 1: not only does it stay there when the state does not change (see Graph 1.2) but it is also attracted to 0 or 1 whenever the money partition gets into the neighborhood of these extremes.

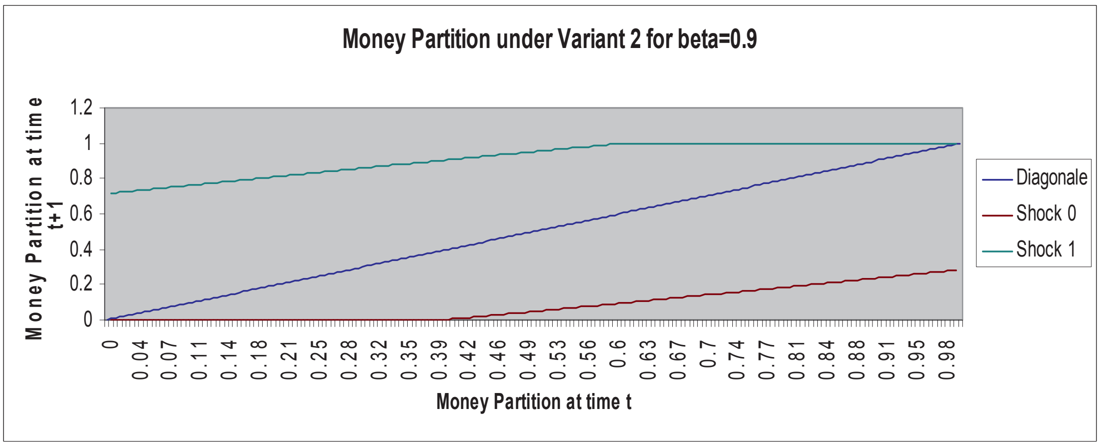


Graph 1.

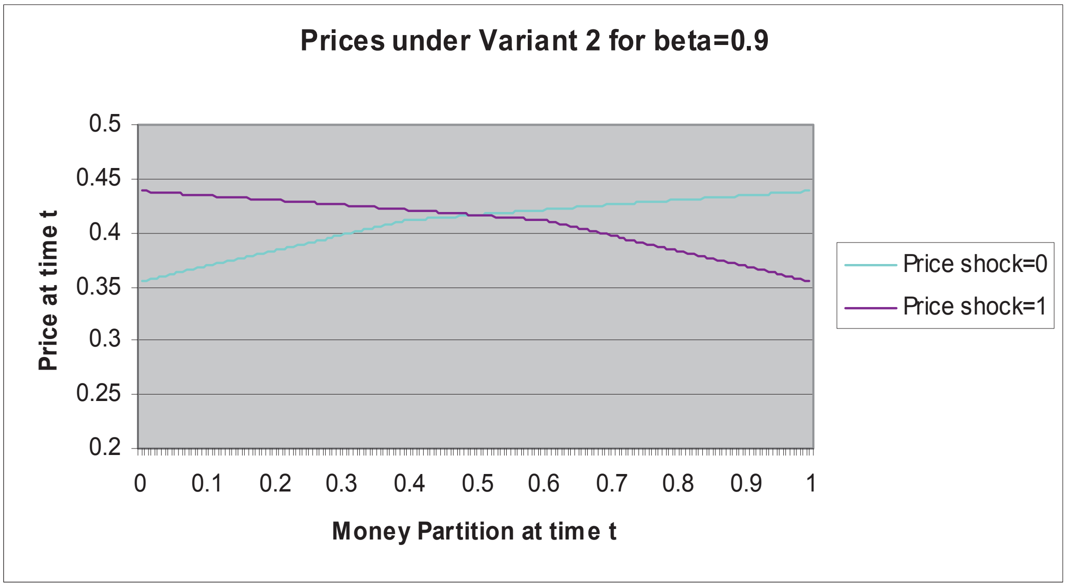


Graph1.2

As evidenced by Graphs 1.1 and 1.2, the link between prices or velocity and the distribution of money across agents is two dimensional whereas its dimension is zero in usual macro models and money demand equations. Moreover, this link is non-linear and non-monotonous in Variant 1. By contrast, in Variant 2, it is continuously increasing or decreasing according to the state and not very far from linear.

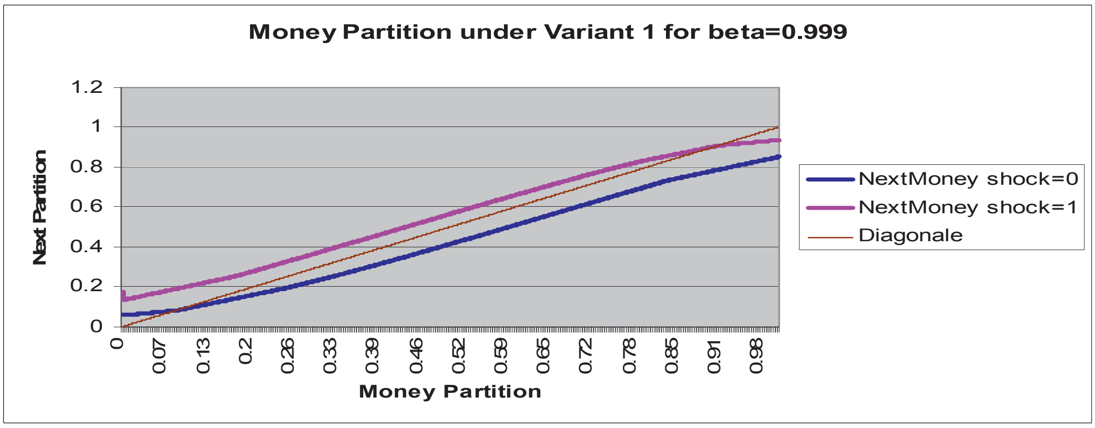


Graph 2.1

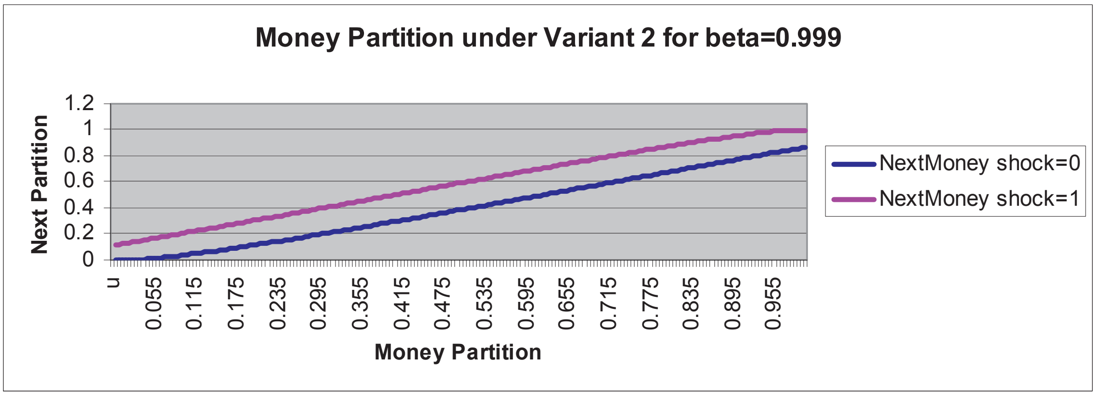


Graph 2.2

Finally, let's look at the case. $\beta = 0.999$. Graphs 1.3 and 1.4 show that the corridors of determination of the money partition become, except for their ends, increasingly similar across the two variants. However, the range $[m_{Min}, m_{Max}]$ which is larger, remains still in the interior of the segment [0, 1].



Graph 1.3



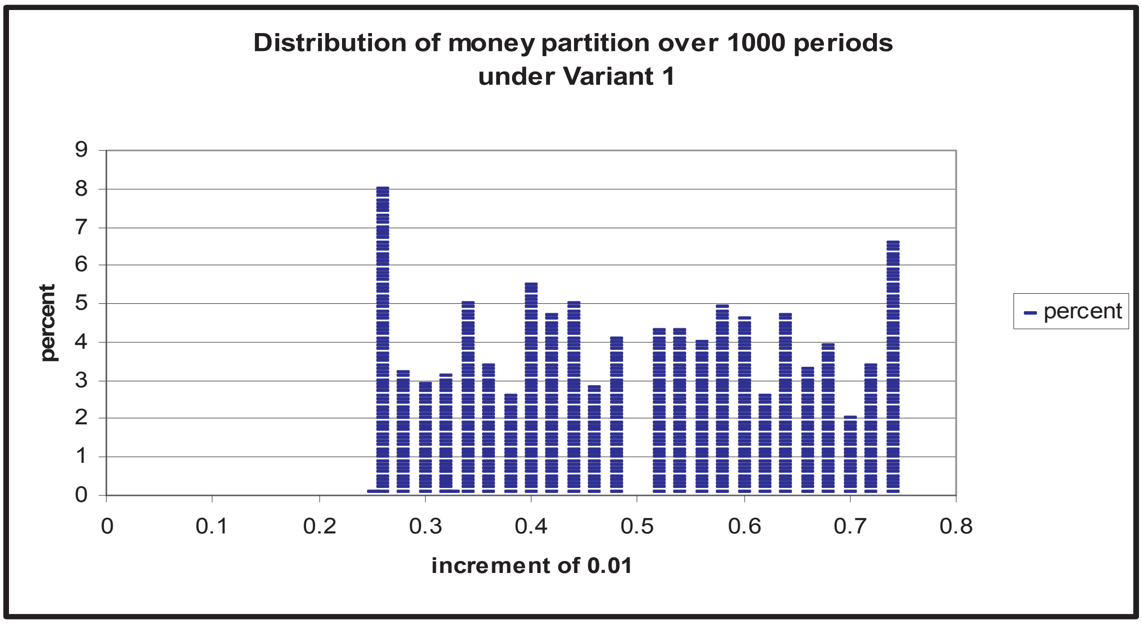
Graph 1.4

<u>The dynamics of money partition across time in Variants 1 and 2</u>

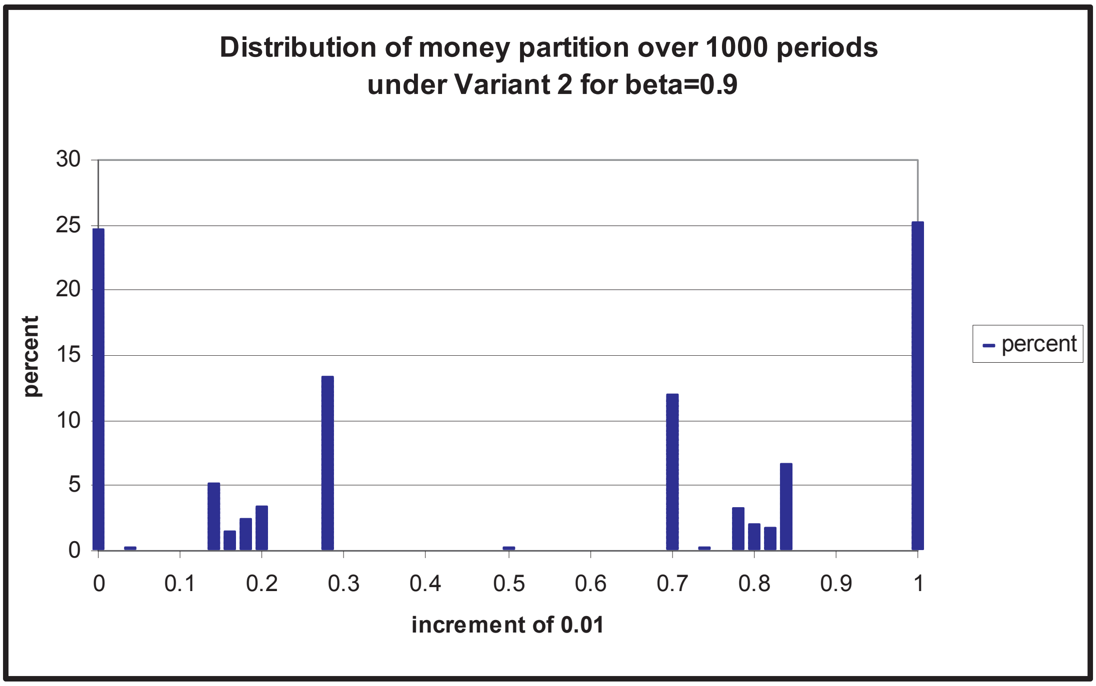
Graph 3.1 and 3.2 show the distribution of money partition under Variants 1 and 2 when simulated over 1000 periods starting with an equal money partition, i.e. $m_0 = 0.5$ at time 0. In both cases, there is some accumulation at the edges of the respective intervals $[m_{Min}, m_{Max}]$ and [0,1].

Graph 3.1 remains similar but not identical for different runs of these 1000 periods. The probability of an equal money partition is always nil if one does not count $m_0 = 0.5$, implying that there is at least one point in the interval $[m_{Min}, m_{Max}]$ and [0,1] which is unlikely to be revisited rapidly. This remark will have interesting consequences in the context of the next section.

Moreover, the occurrences of other values than m_{Min} and m_{Max} are usually lower than for the latter two values. However, the number of such occurrences may vary substantially across runs for high values of β, which explains that some runs were carried out for 2000 periods in order to check that distributions were stabilizing. This makes the behavior of Variant 1 quite different from the behavior of other models in the heterogeneous agents literature (see for instance Imrohoroglu (1992) or Burkhard and Maußner (2004) which explicitly assume invariant wealth distributions. Moreover, the shape taken by the distribution is not consistent with the characteristics of distributions generated by representative agent models as described by Medio (2004). More precisely, the function F in (4.1) should be *topologically transitive*, that is should not only map the respective intervals $[m_{Min}, m_{Max}]$ and [0,1] into themselves but also revisit any point of it after a finite number of periods, which does not seem to be the case.

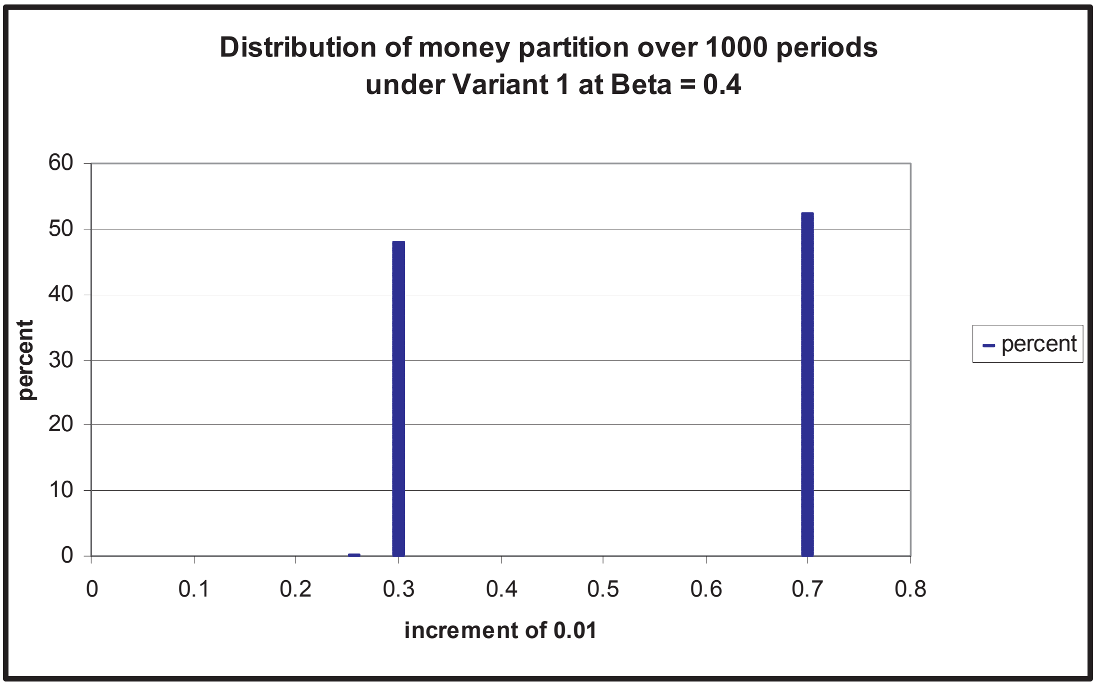


Graph 3.1



Graph 3.2

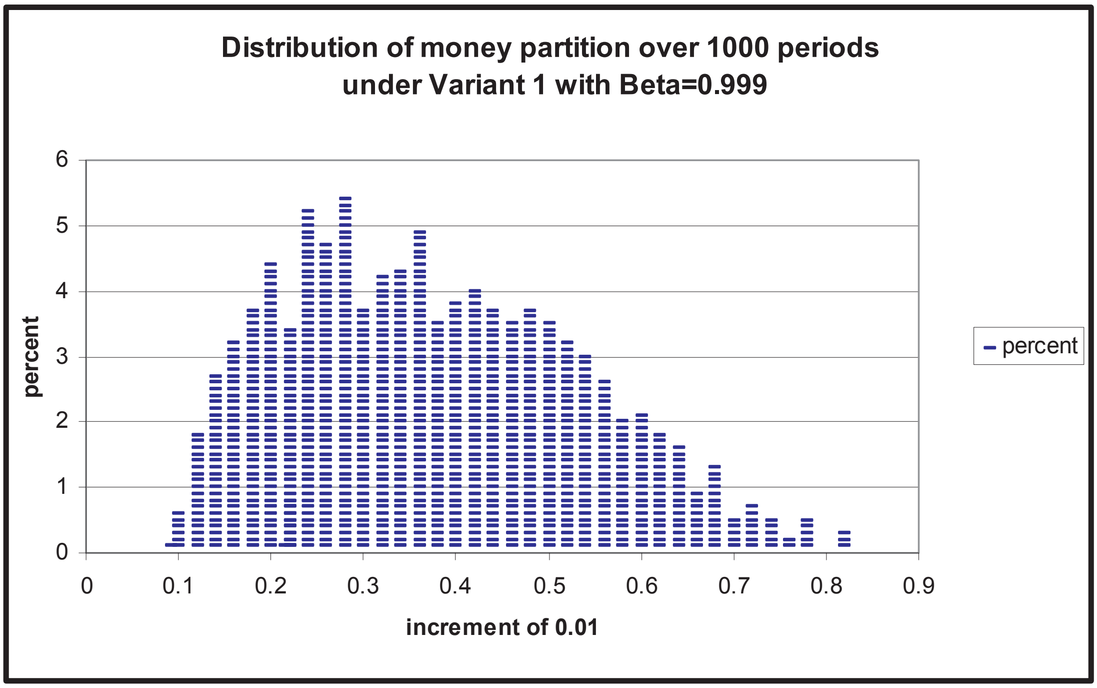
By contrast, Graph 3.2 remains quite stable across runs showing that money partition varies across a limited set of numbers across time in the example chosen and makes discrete jumps across more values, but not so many, than the number of microeconomic shocks. Hence, a rational expectations model with heterogeneous agents can, even without considering investment, create more discrete price and money partition jumps than the number of macro-news. Moreover, it can exhibit more types of jumps than the number of micro-news which create its dynamics. Hence, disequilibrium approaches are not the only possible approaches to a phenomenon that seems to characterize a number of markets, as argued by a number of authors (see for instance Bouchaud (2009) for an interesting and visual summary of this literature).



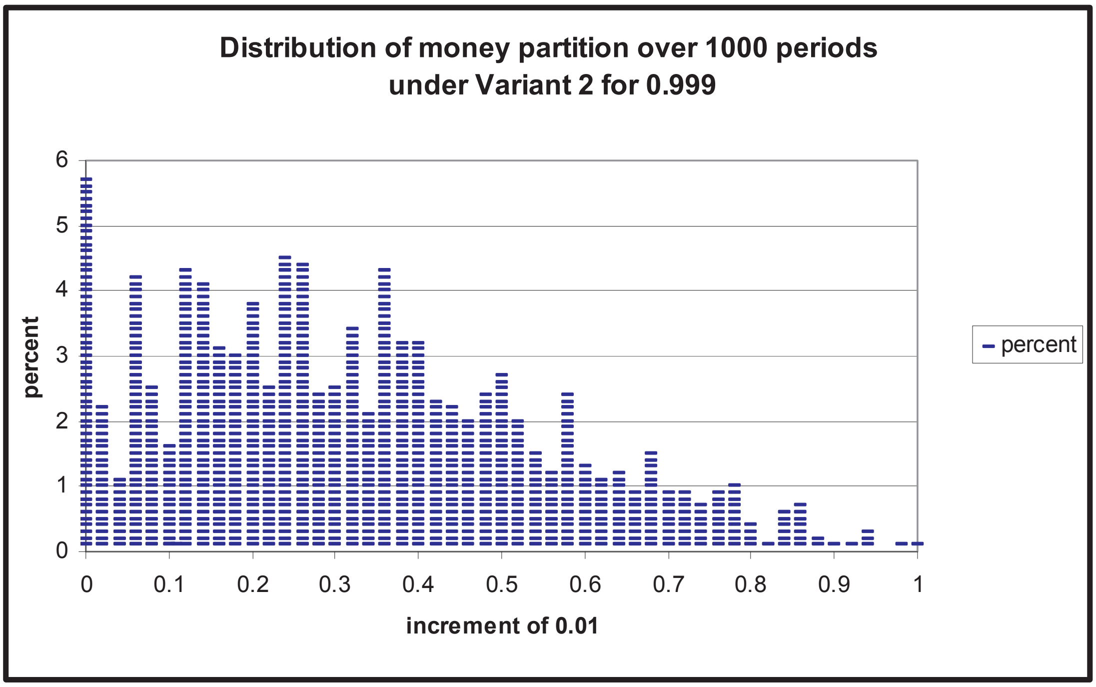
Graph 3.3

Moreover, the nature of the distributions of money partitions generated may vary a lot depending on β. For instance, when β is low and makes cash-in-advance constraints always binding, the partition of money, although variable, always jumps after a few iterations between two values, thereby converging toward an invariant probability distribution, as standard for most general equilibrium models with one agent (Lucas (1980), Lucas and Stokey (1989), Medio (2004)). However, when β increases and becomes close to one, the distribution of the sequence of money partitions becomes more continuous and, under Variant 1, looks increasingly like a normal distribution (see Graph 3.1). But this is not the case under Variant 2 (see Graph 3.4). In that case, once chance has pushed money partition in one direction, it may take a very long time before the other extreme is reached.

Overall, varying β within the context of the general equilibrium models with rational expectations and heterogeneous agents constituted by Variants 1 and 2 allows to create a wealth of dynamics for the partition of money/wealth. As indicated by Farmer and Geneakoplos (2008), some authors (for instance Bouchaud, (2009)) would see them as characteristic of disequilibrium models.



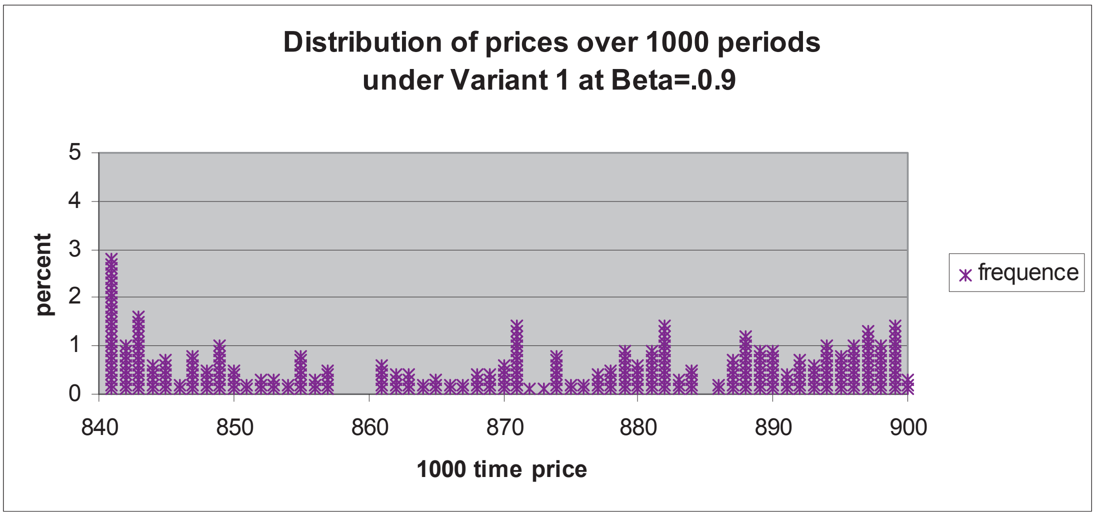
Graph 3.3



Graph 3.4

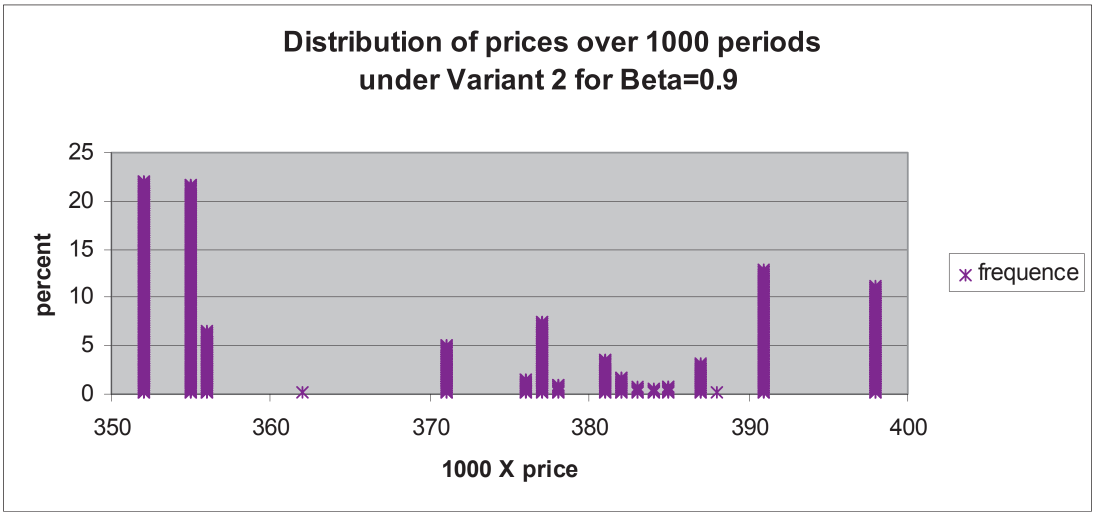
The dynamics of prices across time in Variants 1 and 2

The dynamics of prices is also quite different from the dynamics that could be generated by more traditional models. The assumption of normality of the price distribution is also here deprived of any justification. Moreover, some price jumps are common (Graph 4.1).



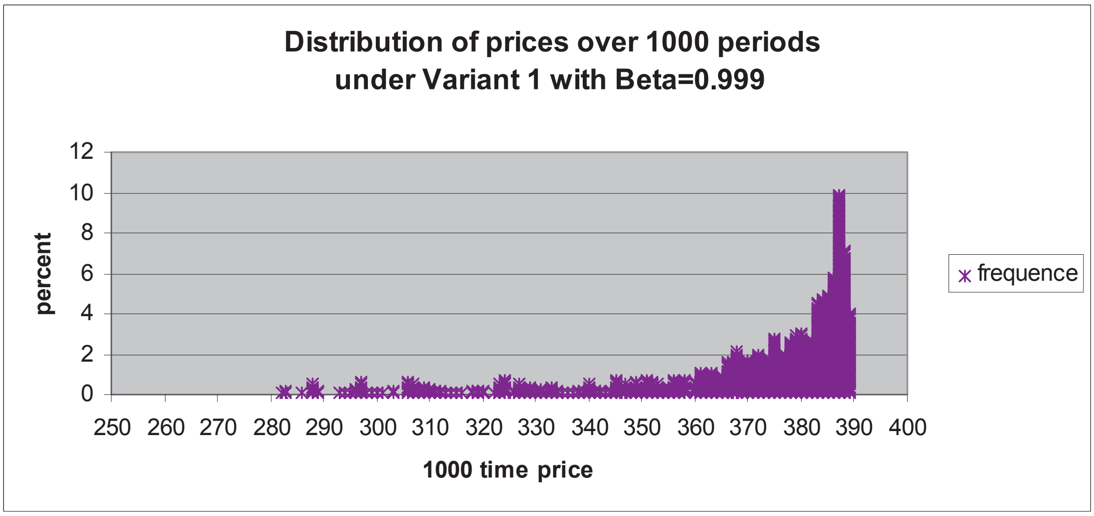
Graph 4.1

As concerns Variant 2, the simulation underlying Graph 4.2 also confirms that jumps across prices accompany the jumps of the money partition.



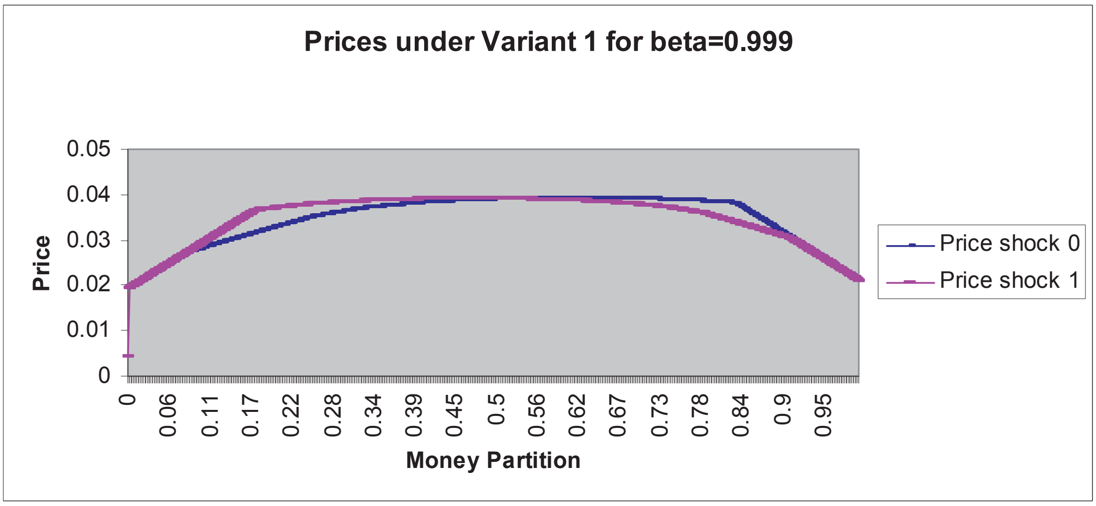
Graph 4.2

As already mentioned above, price jumps can be numerous and be difficult to relate to the number of news, be they macro- or micro-news.

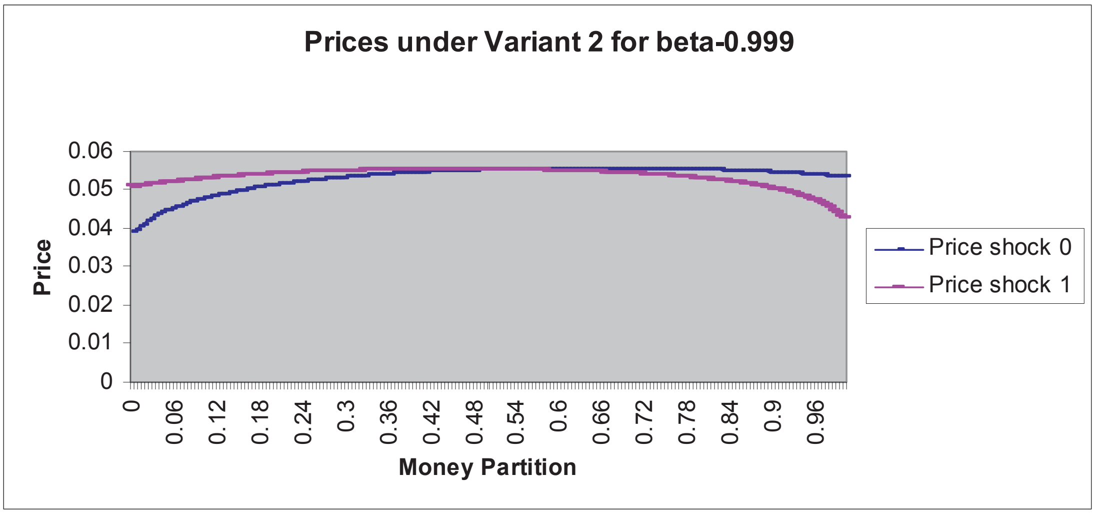


Graph 4''

Looking at the case where $\beta = 0.999$ is again interesting as a distribution similar to a fat tail appears under Variant 1 (Graph 4''). This shape is quite logical as the quasi normality of the money partition under Variant 1 is combined to the symmetrical and close design of the price determination under the two shocks (see Graphs .5 and 5').

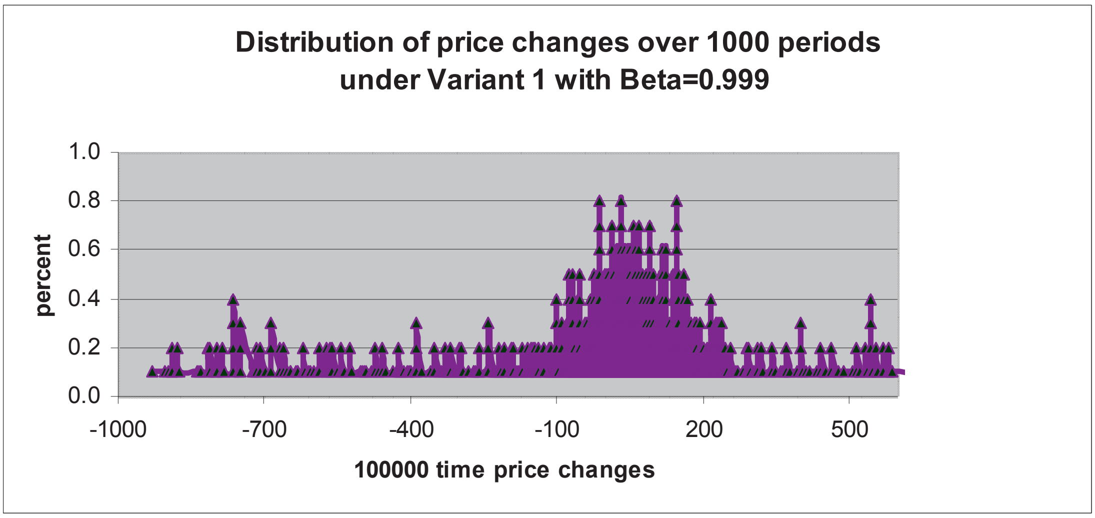


Graph 5.1



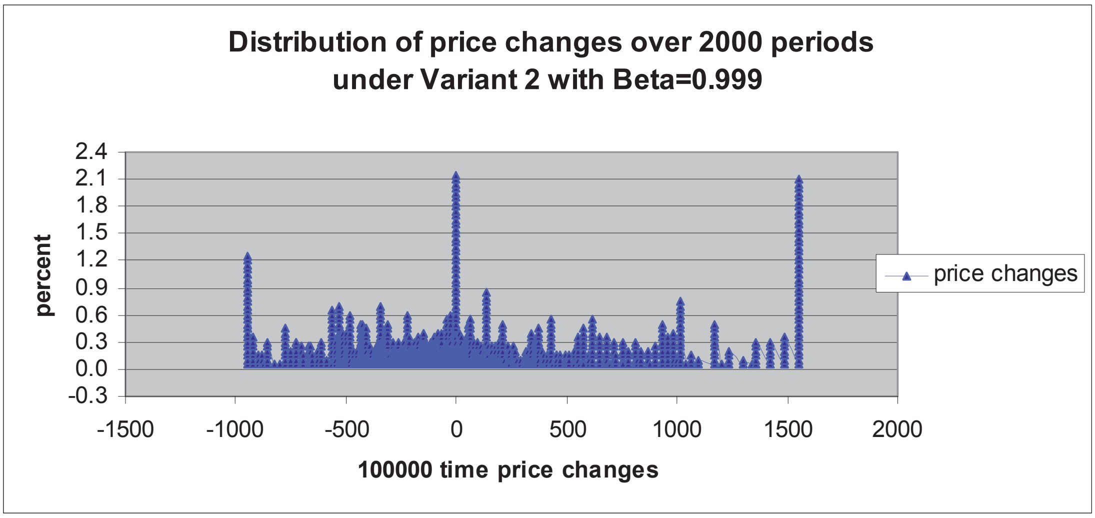
Graph 5.2

This type of fat tail shape also appears when considering the distribution of price changes as under Graph 6 and Graph 6'.



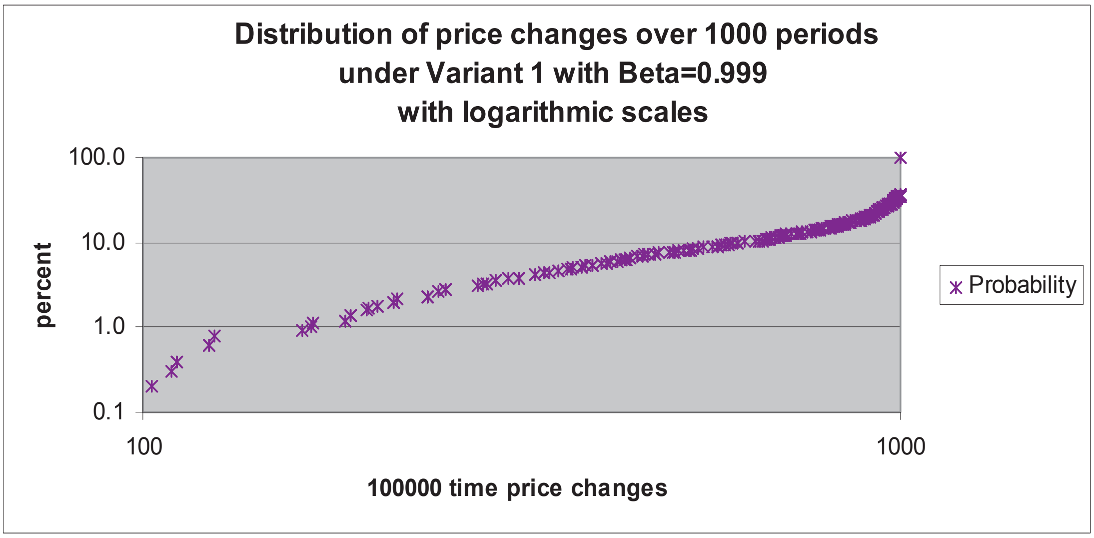
Graph 6.1.

Finally, at such level of the discount rate of utility, it is logical to wonder whether the good traded is not quoted on an almost continuous basis. Hence, it might be interesting to check the shape of the cumulative probability distribution of price changes to check whether it has a specific shape. In particular, one may want to check whether such distribution follows a power-law distribution as found in many real life cases. Although statistical estimation methods have been developed to test such laws, we use here the simplest method available and check it visually by employing logarithmic scales.



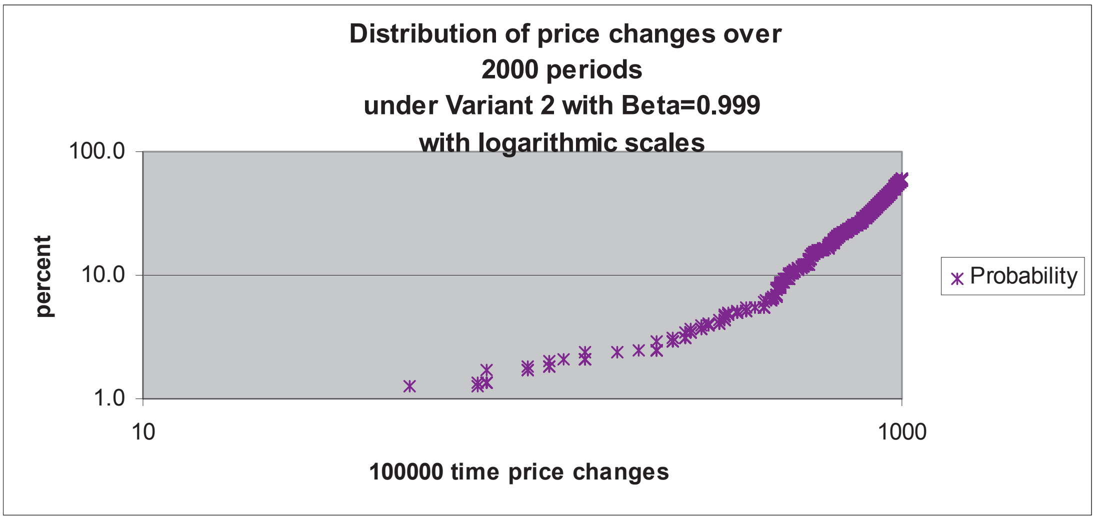
Graph 6.2

As apparent on Graph 7.1, this does not seem to be the case for Variant 1 as the probability always shows some curvature. This might not be really astonishing given that when trading on an organized market, settlements usually take place only at the end of the day at most. Variant 1 might not be best suited to exhibit such characteristic.



Graph 7.1

On the contrary, under Variant 2, some segments of the cumulative probability distribution appear more linear as apparent on Graph 7.2.


Distribution of price changes over
2000 periods
under Variant 2 with Beta=0.999
with logarithmic scales
100.0
10.0
1.0
percent
10
1000
100000 time price changes
Probability

Graph 7

Section 5

MONEY, ITS LEGAL FRAMEWORK, FINANCIAL DEVELOPMENT AND CASH CONTRAINTS

What are the incentives of agents to either keep applying the rules of collective behavior which often implicitly underlie given models or to reject and possibly improve them? This question is rarely asked in general equilibrium models. Indeed, it is often assumed that markets are complete and the associated optimality makes the question irrelevant. Moreover, assuming one representative agent makes it difficult to envisage that diverging interests may jeopardize the structure of the economy and lead to its impairment or to the absence of improvements if needed.

Answering such question is much more natural with two agents whose rational interests may diverge and the possibility of their divergence either lead to the disappearance of markets or prevents their creation. Indeed, our model and its "stationary expectations" equilibrium solutions as described and simulated earlier are associated to several implicit assumptions. First, markets envisaged by both Variant 1 and 2 are functioning as described in Section 2. This implies that agents do not have the possibility to choose between Variants. Second, agents cannot refuse to trade together. Third, they cannot improve any of the two variants by creating new markets, new financial products, or new institutions like banks.

However, to the extent that agents in this model are fully rational and able to make the same calculations as presented in this article, it cannot be excluded that, at any point in time, they compare their current welfare with the level of welfare that they would expect to attain in another model or in another variant. For instance, at any point in time, they may compare their current level of expected welfare with the level which would be reached if they decided at that point in time to restart history and for instance live in autarky forever, like the representative agent from a Lucas-Stokey model with the same endowment and without bond markets. This would constitute *financial regression*. Such comparison may happen at the very start of the day, when agents do not yet know s_t, the shock of the day, or later when s_t is known but purchases are not carried out yet.

Alternatively, if they are living in the Variant 1 model, they may consider whether they would like to jointly create a net clearing system so as to restart history in Variant 2 assuming this change of market rules would be costless. Finally, they may compare their situation with the one they would have in a specific Arrow-Debreu model in which agents are allowed to trade as securities and before the start of history, the same endowment processes they receive in Variants 1 and 2. To the extent that the equilibrium generated by this model is simple enough to be reproduced within our model with the creation of one insurance market, this last model is representative of most possible potential financial developments in the very simplified economy that underlies Variants 1 or 2.

<u>Defining and simulating various experiments</u>

In order to make such comparisons, I simulate the models with the same parameters as in Section 4 and calculate each agent's expected welfare in each of the above cases. Then I discuss the likely preferences of agents depending on the partition of money in order to assess whether they would prefer to restart history in a changed context and, if so, what kind of fine would obviate such preference and hence make the model solution fully consistent .

While models with competitive centralized markets and rational expectations are criticized (or praised) by some for systematically generating systemic risk while promising "Arrow-Debreu type equilibria"", the purpose here is to check whether equilibriums are Pareto-optimal or whether they need to be made so by creating legal incentives. Somehow, this is equivalent to making agents not only participate in one variant of the model but also play a game in which the two agents, having full information on each other may envisage to move together to the Arrow-Debreu model or may decide to move individually to the Autarky model, thus forcing the other agent to a similar move. In such context, identifying legal incentives avoiding financial regression is essential.

This excludes that agents find compromises of their own improving on or avoiding regression to other models, for instance as a result of an "invisible hand" or of a Coase theorem. However, as the Autarky and the Arrow –Debreu model belong to the main alternatives, it is useful to examine how likely shifts to them in a game with a limited set of possible models are. This may either help the "invisible hand" assess the importance of the work to be done in order to suggest an agreement between agents or suggest how the"invisible hand" may itself act if under the shape of a legal authority.

In order to calculate the level of welfare in the economy represented by our model, we estimated the expected utility of each agent at period t before he/she learns about the state of the world s_t and dependant on its wealth as represented by his money holdings:

$$W_i^0(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] \qquad (5.1)$$

We also calculated such expected utility once the agent knows about the state of the world and can anticipate his/her consumption:

$$W_i^1(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}] \qquad (5.2)$$

More precisely, for each of these partitions, we randomly drew 100 processes of 100 random shocks each. Then we calculated the discounted utility generated by each of these processes for each of the two agents evaluated at time t. Then, following a Monte-Carlo approach, we calculated the mean of these 100 utilities in order to obtain an estimate of the expected utility of each agent for each original money partition. Finally, we calculated the economy expected welfare for each of these partitions by simply adding up the expected utilities of the two agents, thereby giving equal weights to each agent. This is done under Variant 1 and Variant 2. We also calculated the expected utilities generated by autarky and our specific Arrow-Debreu economy. Given the simplicity of the stochastic environment chosen, calculations of $W_i^{Au}(m)$ and $W_i^{AD}(m)$ were easy. In the autarky case, each agent may be considered as living in a monetary model similar to ours, but with one representative agent only instead of two. Moreover, each of these representative agents is forbidden to trade with the other representative agent. Hence, in the case of autarky,

$$W_i^{0Au}(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] = \frac{1}{2(1-\beta)}(\log(XSI[i, s_1] + \log XSI[i, s_2]$$

$$W_i^{1Au}(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}]$$

$$= \log XSI[i, s_t] + \frac{\beta}{2(1-\beta)}(\log(XSI[i, s_1] + \log XSI[i, s_2]))$$

In the Arrow-Debreu case, all markets exist and agents can therefore trade their future endowments. As securities on the endowments defined by our choice of Π and XSI can only have equal value at time t before the state in t and in following periods are known, the wealth of our two agents are identical. The equilibrium will therefore generate equal and constant consumption for both agents. Consequently,

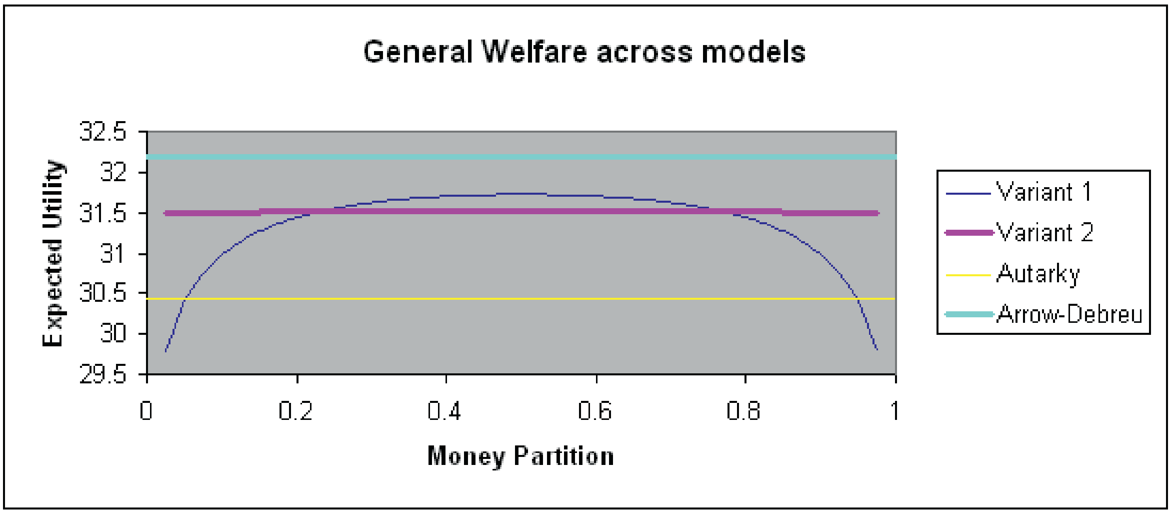
$$W_i^{0AD}(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] = \frac{1}{(1-\beta)}(\log((XSI[i,s_1] + XSI[i,s_2])/2))$$

If however, the agents consider to go to an Arrow-Debreu set up only after they know their current state and their current consumption, they would consume their consumption and

$$W_i^{1AD}(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}]$$

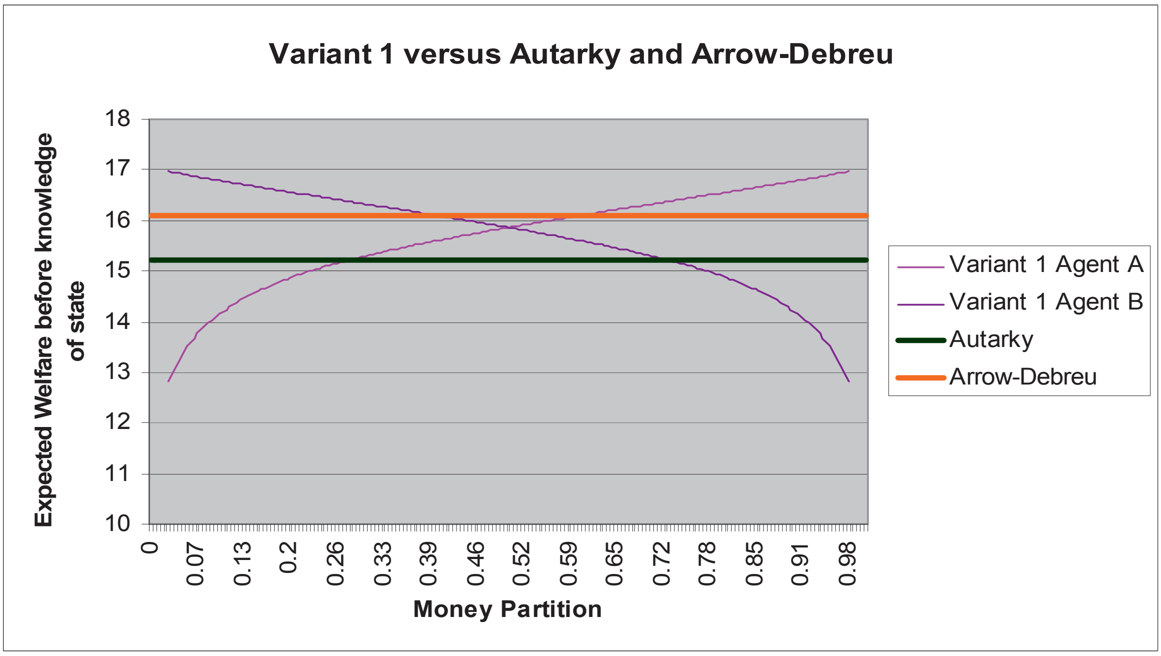
$$= \log XSI[i, s_t] + \frac{\beta}{(1-\beta)}(\log((XSI[i,s_1] + XSI[i,s_2])/2)$$

<u>The Results</u>



Graph 8

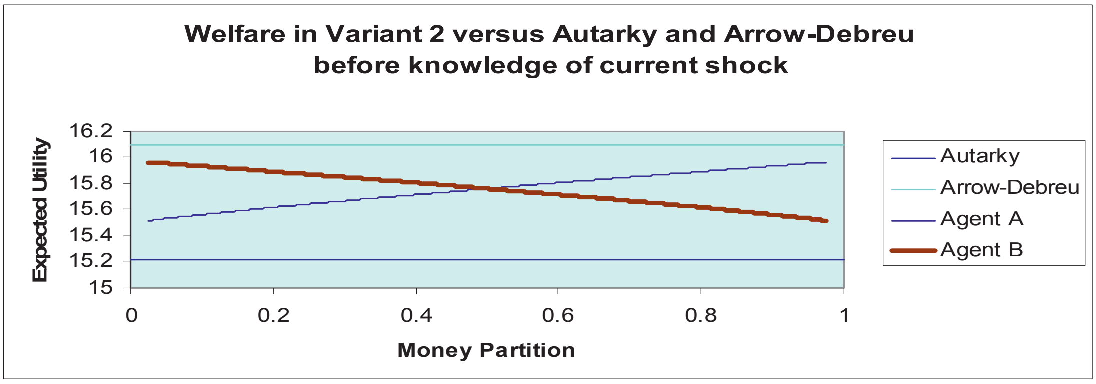
General welfare in the 4 types of economies is presented on Graph 8. Here, the general welfare is calculated by simply adding the expected utility of the two agents. As expected, the Arrow-Debreu model is the best performer. Variant 1 dominates Variant 2 in all cases where money partition has reached its long term range in Variant 1. But Variant 2 dominates when agents start with extreme money partitions. It is important however to note that the interval on which Variant 2 is collectively preferable to Variant 1 does not intersect with the interval $[m_{Min}, m_{Max}]$. Finally the autarky model may be collectively preferable to Variant 1 only for very extreme partitions of money.[13]



Graph 9.1

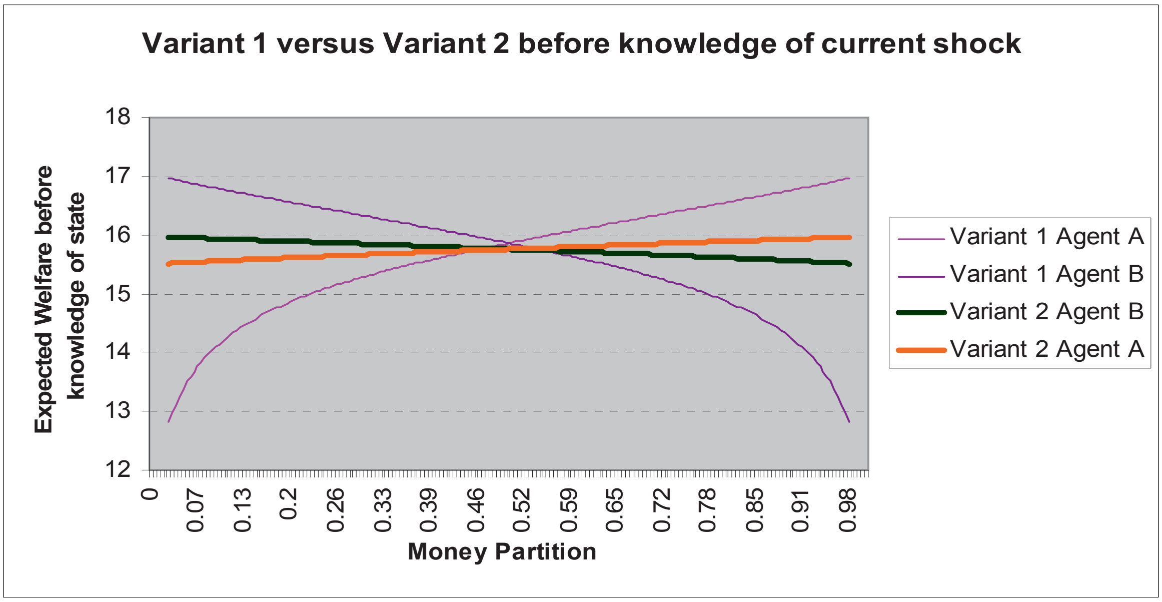
However, considering welfare individually allows for a more precise analysis and leads to different results. In Graph 9.1, two elements stand out. First, in Variant 1, at the beginning of each period before the current shock is known, autarky is preferable to one of the agents whenever the partition of money does not belong to the range [0.28, 0.72]. This range is narrower than the range $[m_{Min}, m_{Max}]$ which is equal to [0.25, 0.75]. Hence, even after the partition has converged to its long term range, it may well reach the range [0.25, 0.28[, where agent A may prefer autarky or the range]0.72,0.75], where agent B may also prefer autarky. On the basis of the probability distribution presented in the previous chapter (Graph 3.1), this could happen in 1 out of 5 periods.

Second, the move to an Arrow-Debreu model, i.e. the creation of a more advanced financial system, will be seen as favorable by both agents only when money partition belongs to [0.4, 0.6], i.e. 40 percent of the time (see Graph 3.1 again or Table 2 below). These elements have consequences both from a legal viewpoint and from a financial development viewpoint as will be seen later. By contrast, these difficulties do not arise with Variant 2 which is always Pareto superior to Autarky and Pareto inferior to Arrow-Debreu (see Graph 9.2). Hence, finding support for financial development in the context of Variant 2 should not be a problem.



Graph 9.2

However, shifting from Variant 1 to Variant 2 or the contrary does not seem so natural, as reflected by Graph 9.3. Whatever the partition of money, one of the agents does not have interest in abandoning the cash-in-advance model for a cash-at-the-end-of the-day model.. Moreover, the likelihood of visiting 0.5, the only money partition where the levels of welfare of the two agents is very close, is very small, as apparent both on Graphs 3 and 3'.

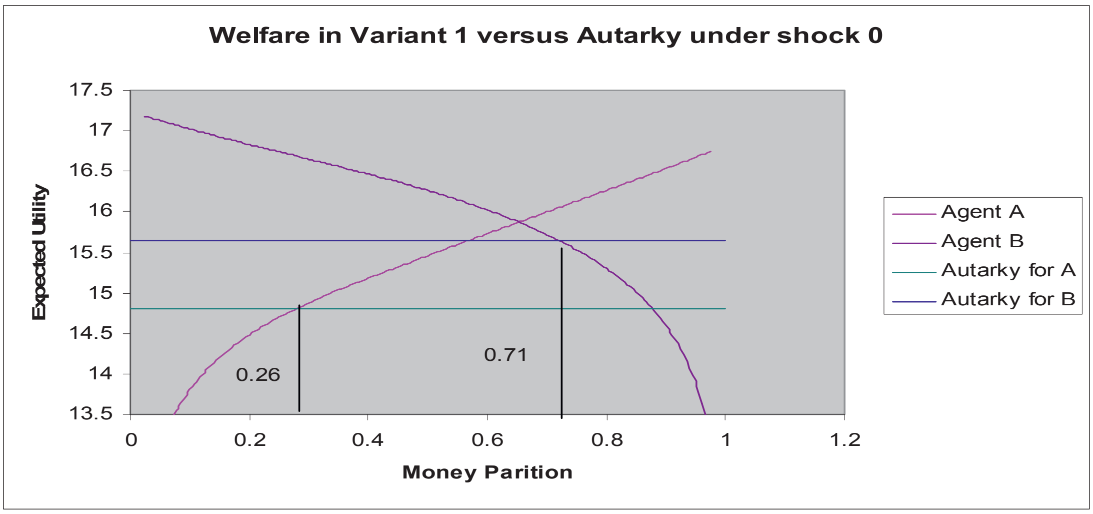


Graph 9.3

Reasoning by analogy, one may identify Variant 1 to gross payment systems and Variant 2 to net payment systems. This leads to the conclusion that it may be difficult, as experienced historically, to replace net payments systems by gross payment systems without official interventions. Alternatively, one may identify Variant 1 to an over-the-counter market and Variant 2 to an integrated centralized market. One may conclude again that it may also be difficult to integrate without official interventions over-the-counter operations into one integrated centralized market.

The legal framework of Variants 1 and 2: Is a Legal Tender status needed?

As apparent in Graph 9.1, for money partitions between 0.25 and 0.28 or between 0.72 and 0.75 and under Variant 1, one of the two agents may prefer autarky to paying back the collective debt represented by money. As apparent in Graph 3, this agent knows that, in a situation where the amount of money he/she owns is particularly low, his endowment might be higher than what his cash constraint will let him/her afford. As a result, the price of the unique good will fall to a low level at which he/she will have to sell anyway. Moreover, this interest for autarky may increase when he/she learn about the state s_t of the day. As apparent in Graph 10, when agent B learns that its endowment in state 0 is high and the money he/she holds is limited, his/her interest is clear in the absence of further incentive: He/she prefers autarky given that the level of income insurance he/she gets from the possession of money is too limited.



Graph 10

Therefore, the effect of introducing money in the economy is not as straightforward as suggested by models like Townsend (1980) or Manuelli and Sargent (1988) who limited themselves to specific money partitions. This phenomenon may explain the non-monetization or demonetization of some economies, either in developing countries or in the European Middle Ages where money partition often became very uneven across agents or regions due to historical developments and was followed by the disappearance of former currencies.

Making the rational equilibrium of Variant 1 sustainable may therefore imply a law on the legal tender status of money or on the functioning of markets[14]. This law needs to affect the incentives of agents to choose between autarky and Variant 1 when the partition of money becomes too unequal. In other words, in order to keep the market in Variant 1 operational, agents must be fined or punished if they refuse to keep participating into the centralized market of Variant 1 The fine or punishment itself must be sufficient in order to have them prefer Variant 1 to autarky without affecting their incentives in terms of trading. The simplest solution is to tax any agent seeking to live in autarky by an amount at least equal, in terms of expected utility, to the difference:

$$\underset{m\subset[0,25,0.28]}{Max}\{W_i^{0Au}(m_t)-W_i^{0}(m_t),\, W_i^{1Au}(m_t,s_t)-W_i^{1}(m_t,s_t)\}=$$

$$\text{Max}\{\frac{1}{2(1-\beta)}(\log(XSI[i,s_1]+\log XSI[i,s_2]-E[\sum_{l=0}^{l=\infty}\beta^l U_i(c_{t+l}^i|I_t],$$

$$\log XSI[i,s_t]+\frac{\beta}{(1-\beta)}(\log((XSI[i,s_1]+XSI[i,s_2])/2)-U_i(c_t^i)-E[\sum_{l=1}^{l=\infty}\beta^l U_i(c_{t+l}^i|I_{t+1}]\}$$

= Max{(15.22261-15.05499); 14.79896-15.47471} = 0.16762, which corresponds to about 1/100 of the overall wealth of the agent..

By contrast, the need for a law on the legal status of money does not manifest itself under Variant 2. This difference across Variants might be an important qualification also to Kiyotaki and Moore, which assumes payment at the end-of-the-day like in Variant 2. Indeed, introducing money under the form of banknotes with a cash-in-

[14] The term "legal tender" may be viewed by some as inappropriate as it usually characterises the payment of debts across agents which in this model do not exist formally.

advance-constraint may be more difficult when a net payment system or a banking system does not facilitate it. It should be recognized however that this supposes the existence of techniques and costs which are not taken into account here.

Financial development and Money

These elements also show that financial development is also dependant on the distribution of money and the degree of equality among agents. Under Variant 1, shifting to the Arrow-Debreu environment is Pareto superior only when the money partition is between 0.4 and 0.6, i.e about 40 percent of the time. Moreover, if making such a reform takes time, the dynamics of money partition decreases the likelihood of a continuous agreement to carry out financial development. In Table 2 below, one calculates how often financial development may happen under continuous Pareto-superiority over 1 to 6 periods. If reforms need 6 periods to be carried out, the likelihood of financial development occurring once over 1000 periods falls to less than ½ percent. Under Variant 2 by contrast, this is always possible.

Table 2

How likely is it that money partition stays N periods in a row between 0.4 and 0.6? (in percent)					
N=1	N=2	N=3	N=4	N=5	N=6
39.70%	8.30%	3.60%	1.70%	0.90%	0.40%

.

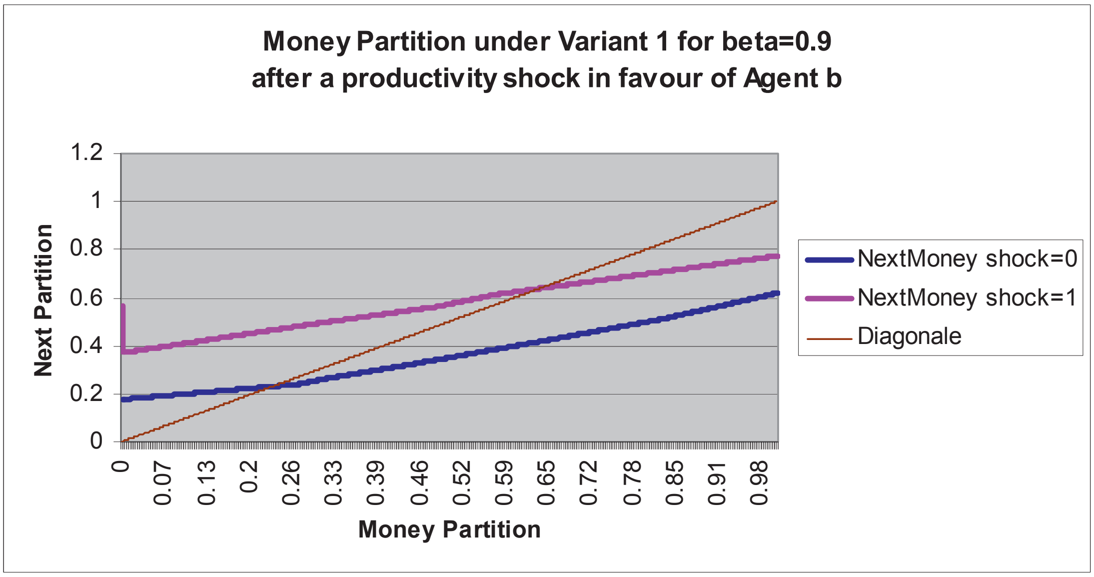
However, shifting from Variant 1 to Variant 2 or the reverse is not easy either, as noticed before. Therefore, in a heterogeneous world with monetary frictions, excessive inequalities and variability in the partition of money may delay financial development. As the range of money partitions where our specific Arrow-Debreu equilibrium is Pareto-superior to the cash-in-advance equilibrium is limited, barring any redistribution of ownership rights, it may often be difficult for authorities of a given country to gather the political support necessary to open new financial markets: such markets make obsolete the advantages of some of the country agents.

In that context, it may be useful to note that the creation of net clearing techniques as available under Variant 2 may appear from a collective viewpoint as more valuable than reflected by the calculation of general welfare in Graph 8. Variant 2 is valuable not only because it diffuses systemic risk while avoiding the need for legislation but also because it accelerates financial development. By contrast, the creation of a legal incentive to stabilize Variant 1 may be suboptimal from a long term prospective if it makes financial development too uncertain. We will come back to this issue after dealing with systemic risk in the next subsection.

<u>Systemic risk and money in the absence of banks and other financial markets</u>

Under Variant 1, an absence of legal tender status for the currency may lead to the disappearance of the corresponding equilibrium. This implies also that, should some characteristic of Variant 1 evolve unexpectedly, it might be important to adapt this legal status and, in particular, the fines avoiding attempts by the two agents to go to autarky. This allows the joint set of models considered, i.e Variant 1, Autarky and Arrow-Debreu to generate some systemic risk even though in each of them, agents have rational expectations.

Systemic risk exists when the occurrence of a given event has the consequence that the functioning of the economy is durably altered and overall welfare is durably reduced (see Aglietta and Moutot (1993) and CGFS(2010)) This can only happen in our suite of models if one of the agents prefers autarky and cannot be discouraged from doing so by an appropriate legal constraint.

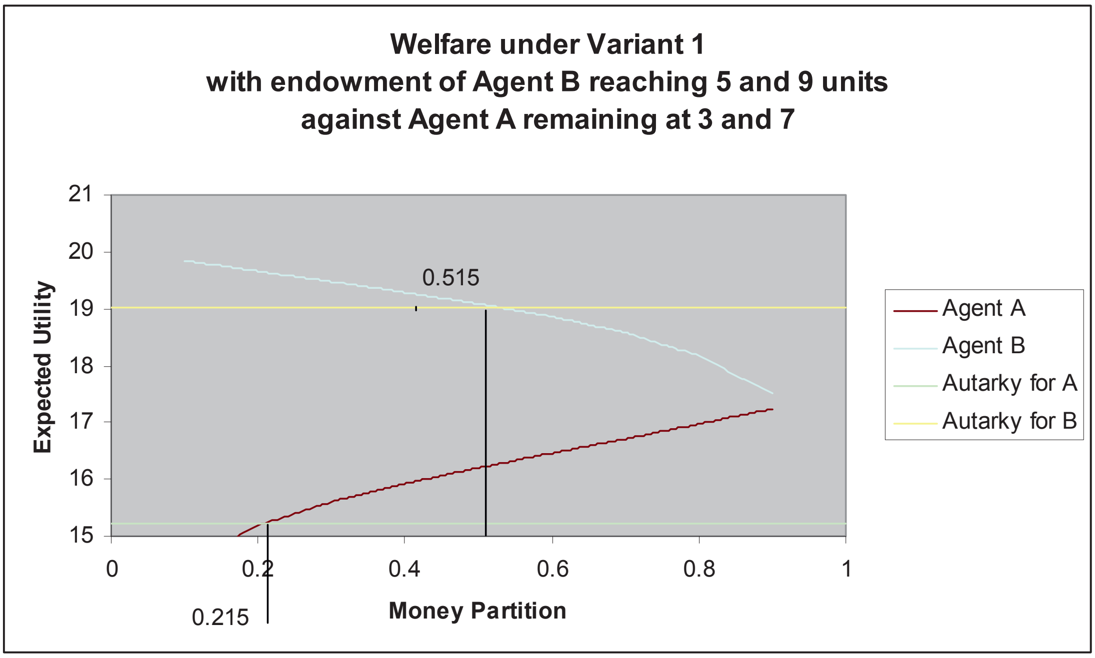


Graph 11

Let us therefore assume that the economy is well described by Variant 1 as defined above and that a legal tender status envisages fines up to 0.16762 units of utility making this cash-in-advance economy perfectly stable and rational under its usual endowments.

But let us imagine also that at a given point in time, while the partition of money is close to 0.5, the endowments of one of the two agents, agent B, unexpectedly shifts from alternating between 3 and 7 units of goods to alternating between 5 and 9. As shown in Graph 11, this leads to a new equilibrium in which money partition starts oscillating between two new values of m_{Min} and m_{Max} now forming the range [0.23, 0.64] which is narrower than the preceding range [0.25, 0.75].

Let us assume that the authorities do not adjust immediately the fines associated to the legal tender status. If money partition reaches 0.64, the utility of agent B will switch from 18.7449 to 19.03331 if he/she shifts to autarky, implying that a higher fine, precisely a fine of at least 0.288408 units of utility is needed to protect the economy against systemic risk. If the fine however remains limited to 0.16762 units of utility, the temptation to switch to autarky will exist for Agent B for all money partitions in the range [0.515, 0.64] as apparent from Graph 12. A systemic event may therefore happen about 14% of the time and lead to an overall average loss of general welfare of 0.938788.



Graph 12

Hence, a systemic risk may appear and manifest itself in a model with centralized markets and rational expectations whenever financial frictions create the need for a legal environment and this legal environment does not evolve in due time, i.e. does not adjust quickly enough to unexpected changes of the endowment process. Let us note moreover that this example increases the inequality of endowments across agents from 7/3 to 9/3 and that such inequality is directly related to the need to strengthen regulation. This example may thus provide a practical and relatively simple illustration of the link made by Rajan (2010) between inequality and systemic risk in his book called "Fault Lines".

Systemic risk as an occasion to foster financial development

Finally, it may be interesting to clarify that a systemic event may be an occasion to foster financial development, if used adequately. Indeed, suppose that the economy is under Variant 1 and therefore chances that financial development proceeds are as described by Table 2, i.e. financial development is very unlikely. It may be rational for a government to refuse repairing the legal system as suggested above.

Indeed, if the authorities only offer to their two types of agents the possibility of a step to Variant 2 instead of a repair of the legal framework of Variant 1, the immediate level of expected welfare of the economy will in aggregate be immediately reduced as apparent on Graph 8. But the chances of improving this less efficient economy in the time needed to shift from Variant 2 to the Arrow-Debreu model will increase to 100%. The overall loss of welfare generated by a shift from Variant 1 to Variant 2 is much smaller than the benefit of shifting in a few periods to the Arrow-Debreu, as apparent on Graph 9.2. Hence, such a fall is, in the end, preferable to a return to the low likelihood of financial development reflected by Table 2. One could even think that mischievous or Machiavellian authorities would be glad to let systemic risk occur whenever they are sure of their ability to move in a short period of time from Variant 1 to Arrow-Debreu through Variant 2. This is in line with a Schumpeterian view of the world: as crises are necessary to development, no crisis should be wasted.

Conclusion

Two versions of a small model with two agents, money, rational expectations and non-continuously binding cash constraints have been presented and simulated. Together they show that the introduction of heterogeneous agents and cash constraints in DSGE models may play an important role in helping answer questions like: How does the introduction of financial frictions affect the link between money and prices? Also, does it affect financial development? In particular, it makes it possible to consider and describe systemic risk in the context of a model with rational expectations. Systemic risk appears as the risk that agents may want to shift across general equilibrium models as a result of Pareto-comparisons. Hence, the reason for the risk of a systemic event is identified although no full description of systemic events is provided as the model does not consider precisely how the shift from equilibrium to the next actually takes place across time.

This approach constitutes in my view a relatively sound basis to model such specific events as systemic events. Having clarified the incentives for shifts in case they are unexpected and agents are rational, making other hypotheses, in particular on the role of behavioral characteristics in the short term when such events take their actual shape would in my view become natural and logical. Indeed, starting a study of systemic risk with assuming rational expectations does not imply that the study should stop there and that agents should necessarily and always be deemed rational or financial frictions other than cash constraints should be neglected. Other assumptions may have to be considered in parallel to rational expectations. However, rational expectations should remain as a reference, all the more that, as shown above, they can in combination with heterogeneity generate both highly variable distributions of wealth as well as systemic risk.

Finally, because an approach making reference to rational expectations on the one hand and distinguishing "normal equilibriums" from "financially-developed" or "regressive" equilibriums creates the possibility to separate systemic events from their genesis, it creates many more possibilities for underlying assumptions to be proven wrong through both studies of long term datasets of wealth and price dynamics and studies of systemic events. Because this approach is therefore more easily refutable or, to use the terminology of Karl Popper (1934) *"falsifiable"*, it is more likely in my view to support joint and faster progress in the studies of systemic risk and of macro-economic models.

A long way remains however before macro-economists may use such models for practical purposes. Obviously, the mathematical and computational difficulties associated to such models are important and largely condition their use. The combination of heterogeneity with frictions makes the issue of compactness of the corresponding functional operators and of the existence of their fixed points essential and at the same time very complex. Moreover, such models also generate questions on how to deal rationally with complexity and the difficulty for agents to solve and use such models.

However, such study may help in many fields of high relevance to policy makers. Legal frameworks necessary to obviate systemic risks and more generally support financial stability may be better assessed with recourse to such type of modeling. Financial development/regression also may benefit as well as the associated regulatory issues. More generally, modeling together heterogeneity and financial frictions to measure their impact on the transmission process of monetary policies depends on the mastering of techniques associated to such models. One may therefore wonder whether a special effort should not be made, for instance by central banks, in view of supporting their development. In my view, even such simple models as the one presented in this paper can go some way in integrating the views and experience of the so-called "econo-physicists" into macro-economics.

Box 1

Why use cash-in-advance and cash-at-the-end–of-the-day constraints when a number of economists have seriously questioned their use?

Arguably, the models which offer the best micro-foundations to the use of money are mostly based on matching and search theory. For instance, it is sometimes alleged that cash-in-advance constraints would have weak micro-economic foundations as they would only detract from an efficient equilibrium while the matching and search approach would show how money can only elevate the welfare of society. Also, it is often mentioned that models, while having sound micro-economic foundations, should be easily tractable (Wallace and Wright 2009). Indeed, even the simplest cash constraint, the cash-in-advance constraint "à la Clower" (1967) leads to non-linearity that make models difficult to solve.

However, search theory usually includes reference to islands rather than the actual environment of our economies and addresses long-term rather than short-term developments in finite period models. It is therefore useful to consider models with the ability to produce and match time series like those with infinitely lived agents and cash constraints. Also, many turnpike or island models are very similar to cash-in-advance models. The model in this paper for instance is a stochastic generalization of the Townsend (1980) turnpike model. Moreover, in most modern societies, the most obvious feature of money is that money conditions the purchase of goods or services and the payback of debts. Cash constraints express exactly this fact and modern macro-theory faced with repeated financial crises cannot anymore ignore the importance of liquidity constraints. Furthermore, making theory dependant on mathematical simplicity rather than its ability to match reality is highly debatable. If all models were both mathematically tractable and a good description of reality, would the calculation of all possible contingencies not be easily done and therefore taken into account by authorities? Hence, could systemic risk exist? Moreover, several equations used in physics still have no general solutions but are still considered relevant. So why not accept their use in economics?

Indeed, while other modeling strategies have helped consider issues related to the inter-action between money and other public paper or between money and the need for banks or clearing systems, they have not allowed for the creation of infinite time series or are also associated with controversial assumptions. Kocherlakota who has shown the need for money in order to keep memory of past developments, assumes shocks to liquidity preferences (Kocherlakota (2002)), when envisaging such issues. Freeman (1995) considers clearing systems, non-continuously binding constraints and their consequences for monetary policy but uses an overlapping-generations model forbidding the contemplation of short-term dynamics.

Finally, the issue of the respective role of money and bank credit is confronted by Kiyotaki and Moore (2008), who argue that bank credit would be in principle, more efficient than money as a means of transaction, were it not for credit risk and propose interesting explanations to some interest rates puzzles under the assumption that markets are complete. However, they thereby renounce to explain financial development. Their modeling strategy does not allow them to examine under which conditions a well integrated economy with money and price-taking rather than bargaining, with no banks or financial markets, would want to add financial markets, and/or clearing systems to money. Moreover, they do not prove the existence of a solution to the cash-at-the-end-of-the day constraints they use. By contrast, Hansen and Imrohorglu (1992) followed by a large literature summarized by Hausner and Heer (2004), introduce populations of heterogeneous and infinitely-lived agents with cash constraints. But they use them to see how their distribution allows their overall behavior to result in stationary macro-economic time-series after iterations which we do not study.

Overall, using cash constraints to study issues like money demand or money velocity stability, financial development or regression, their accompanying legal apparatus and the occurrence of systemic risk is, in my view, a defendable strategy.

Annex 1

Price regimes in the Cash in advance and Cash-at-the end-of-the-day variants of the model

Variant 1: some characteristics of solutions

The following theorem and lemmas were proved in Moutot (1991) in Appendix C. By contrast, proofs concerning Variant 2 are new.

Lemma 1

Be h $= h(m, \theta^{a}, \theta^{b}, \xi)$. Then the following propositions are equivalent.

$\theta^a \le \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \le \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_1

$\theta^a \ge \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \ge \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_4

$\theta^a \ge \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \le \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_3

$\theta^a \le \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \ge \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_2

<u>Proof</u>

Suppose $\theta^a \le \frac{1}{h} U_a'(\frac{m}{h})$. Then, as $U_a'(.)$ is decreasing, it is equivalent to

$$\frac{m}{h} \le U_a^{'-1}(\theta^a h)$$

and therefore

$$\frac{1}{h} \le \frac{1-m}{h} + U_a^{'-1}(\theta^a h)$$

$$\frac{m}{h} + U_b^{'-1}(\theta^b h) \le U_a^{'-1}(\theta^a h) + U_b^{'-1}(\theta^b h)$$

As a consequence of (1.8) and (1.10), these two inequalities imply that:

$$\theta^a \le \frac{1}{h} U_a'(\frac{m}{h}) \Leftrightarrow h = h_1(\xi) \text{ or } h = h_2(m, \theta^b, \xi)$$

On the reverse, suppose $\theta^a \geq \frac{1}{h}U_a'(\frac{m}{h})$. By the same token,

$$\theta^a \geq \frac{1}{h}U_a'(\frac{m}{h}) \Leftrightarrow h = h_3(m,\theta^a,\xi) \text{ or } h = h_4(\theta^a,\theta^b,\xi)$$

By symmetry, the same manipulation of inequalities show that

$$\theta^b \leq \frac{1}{h}U_b'(\frac{1-m}{h}) \Leftrightarrow h = h_1(\xi) \text{ or } h = h_3(m,\theta^a,\xi)$$

$$\theta^b \geq \frac{1}{h}U_b'(\frac{1-m}{h}) \Leftrightarrow h = h_2(m,\theta^b,\xi) \text{ or } h = h_4(\theta^a,\theta^b,\xi)$$

The two by two combinations of these four inequalities prove lemma 1.

Lemma 2

Be θ^a, θ^b, and ξ three strictly positive real numbers. Be m a real number in [0,1]. Then the two following propositions are equivalent:

$$\theta^a \leq \xi U_a'(m\xi) \text{ and } \theta^b \leq \xi U_b'((1-m)\xi) \Leftrightarrow h(m,\theta^a,\theta^b,\xi) = h_1(\xi) = \frac{1}{\xi}$$

<u>Proof</u>

Suppose first that: $\theta^a \leq \xi U_a'(m\xi)$ and $\theta^b \leq \xi U_b'((1-m)\xi)$.

By definition of h,

$$h(m,\theta^a,\theta^b,\xi) \leq h_1(\xi) = \frac{1}{\xi} \qquad \text{for all m in [0,1]}$$

Given that $U_i'(.)$ is strictly decreasing,

$$U_a'^{-1}(\theta^a h_1) \geq m\xi \text{ and } U_b'^{-1}(\theta^b h_1) \geq (1-m)\xi$$

Consequently, $\frac{m}{h_1} + U_b'^{-1}(\theta^b h_1) \geq \xi$

$$\frac{1-m}{h_1} + U_a'^{-1}(\theta^a h_1) \geq \xi$$

and $\qquad U_a'^{-1}(\theta^a h_1) + U_b'^{-1}(\theta^b h_1) \geq \xi$

Hence, by definition of h in (1.8), $h(m,\theta^a,\theta^b,\xi) = h_1(\xi) = \frac{1}{\xi}$ for all m in [0,1].

Reciprocally, suppose that $h(m,\theta^a,\theta^b,\xi) = h_1(\xi) = \frac{1}{\xi}$.

By definition of h, $\frac{m}{h_1}+U_b'^{-1}(\theta^b h_1)\geq\xi$ and $\frac{1-m}{h_1}+U_a'^{-1}(\theta^a h_1)\geq\xi$.

Thse two inequalities again imply that:

$$\theta^a\leq\xi U_a'(m\xi) \text{ and } \theta^b\leq\xi U_b'((1-m)\xi). \text{ Q.E.D.}$$

Theorem 1

Be U_a (.) and U_b(.) two continuously differentiable, strictly increasing and strictly concave utility functions. Be β strictly superior to 0.

If there exists "a stationary expectations equilibrium" and if it is such that, after a finite number of periods, both agents are always under cash constraint, then:

$$\beta\leq\beta_{\max}=\underset{\substack{s\in S\\ s''\in S\\ i\in\{a,b\}}}{Min}\frac{\xi(s)U_i'(\frac{\xi^i(s'')}{\xi(s'')}\xi_i(s))}{\int_S\xi(s')U_i'(\xi^i(s)\frac{\xi(s')}{\xi(s)})\pi(s,ds')},(3.1)$$

<u>Proof</u>

Suppose both agents are under cash constraint. Then

$$\text{m'}=M(m,\theta^a(s,m),\theta^b(s,m),\xi(s))=\frac{\xi^a(s)}{\xi(s)}$$

which implies that, save for time 0 possibly, m always belongs to δ, the set of values $\{\frac{\xi^a(s'')}{\xi(s'')}\}$ for $s''\subset S$

Lemma 1 implies that

$$Max\{\theta^a(s',m'),\frac{1}{h'}U_a'(\frac{m'}{h'})\}=\xi(s')U_a'(\xi(s')\frac{\xi^a(s)}{\xi(s)})$$

$$Max\{\theta^b(s',m'),\frac{1}{h'}U_b'(\frac{1-m'}{h'})\}=\xi(s')U_b'(\xi(s')\frac{\xi^b(s)}{\xi(s)})$$

However, $\theta(s,m)$ being a solution to (1.13), this implies that:

$$\theta^i(s,,m)=\beta\int\xi(s')U_a'(\xi(s')\frac{\xi^a(s)}{\xi(s)})\pi(s,ds')\quad\forall(s,m)\subset SX\delta,\quad\forall i\in\{a,b\}.$$

However, Lemma 2 implies that if cash constraints are to be binding for all m in (0,1),

$$\theta^i(s,m) \le \xi(s) U_i\,'(\frac{\xi^i(s''}{\xi(s'')}\xi(s)) \quad \forall i \in \{a,b\}, \forall s, \subset S, \text{ and for all } s'' \subset S.$$

Therefore,

$$\beta \le \beta_{\max} = \underset{\substack{s \in S \\ s'' \in S \\ i \in \{a,b\}}}{Min} \frac{\xi(s) U_i'(\frac{\xi^i(s'')}{\xi(s'')}\xi(s))}{\int_S \xi(s') U_i'(\xi^i(s)\frac{\xi(s')}{\xi(s)})\pi(s,ds')}$$

Variant 2 solutions: Some characteristics

Lemma 1'

$\theta^a \ge \frac{1}{h} U_a\,'(\frac{m}{h} + \xi^a)$ and $\theta^b \ge \frac{1}{h} U'_b\,(\frac{1-m}{h} + \xi_b) \Leftrightarrow h$ under h_4

$\theta^a \ge \frac{1}{h} U_a\,'(\frac{m}{h} + \xi^a)$ and $\theta^b \le \frac{1}{h} U'_b\,(\frac{1-m}{h} + \xi_b) \Leftrightarrow h$ under h_3

$\theta^a \le \frac{1}{h} U_a\,'(\frac{m}{h} + \xi^a)$ and $\theta^b \ge \frac{1}{h} U'_b\,(\frac{1-m}{h} + \xi_b) \Leftrightarrow h$ under h_2

Proof

As $U_i^{'}(.)$ is by assumption decreasing,

$\theta^a \ge \frac{1}{h} U_a\,'(\frac{m}{h} + \xi^a)$ and $\theta^b \ge \frac{1}{h} U'_b\,(\frac{1-m}{h} + \xi_b) \Leftrightarrow$

$U_a^{'-1}(\theta_a h) \le \frac{m}{h} + \xi^a$ and $U_b^{'-1}(\theta_b h) \le \frac{m}{h} + \xi^b \Leftrightarrow$

$U_b^{'-1}(\theta_b h) + \frac{m}{h} + \xi^a \ge U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h) \le \frac{m}{h} + \xi^b + U_a^{'-1}(\theta_a h)$

$\Leftrightarrow h$ under h_4 as a result of (1.8'). Similarly,

$\theta^a \ge \frac{1}{h} U_a\,'(\frac{m}{h} + \xi^a)$ and $\theta^b \le \frac{1}{h} U'_b\,(\frac{1-m}{h} + \xi_b) \Leftrightarrow$

$U_a^{'-1}(\theta_a h) \le \frac{m}{h} + \xi^a$ and $U_b^{'-1}(\theta_b h) \ge \frac{m}{h} + \xi^b \Leftrightarrow$

$$U_b^{'-1}(\theta_b h) + \frac{m}{h} + \xi^a \geq \frac{m}{h} + \xi^b + U_a^{'-1}(\theta_a h) \leq U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h)$$

$\Leftrightarrow h$ under h_3 as a result of (1.8'). Finally,

$$\theta^a \leq \frac{1}{h} U_a'(\frac{m}{h} + \xi^a) \text{ and } \theta^b \geq \frac{1}{h} U'_b(\frac{1-m}{h} + \xi_b) \Leftrightarrow$$

$$U_a^{'-1}(\theta_a h) \geq \frac{m}{h} + \xi^a \text{ and } U_b^{'-1}(\theta_b h) \leq \frac{m}{h} + \xi^b \Leftrightarrow$$

$$\frac{m}{h} + \xi^b + U_a^{'-1}(\theta_a h) \geq \frac{m}{h} + \xi^a + U_b^{'-1}(\theta_b h) \leq U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h)$$

h under h_2 as a result of (1.8').Q.E.D.

Proposition A.1

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$.

If $h = h_2(m, \theta^b, \xi^{b)} = \frac{1}{\xi^b}(m + \frac{1}{\theta^b})$, then $m \leq \frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a}$ and $M(m,\theta,\xi) = 0$.

If h= $h_3(m, \theta^a, \xi^{a)} = \frac{1}{\xi^a}(1 - m + \frac{1}{\theta^a})$, then $m \geq 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a}$ and $M(m,\theta,\xi) = 1$.

If h= $h_4(m, \theta^a, \xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b})$, then

$$\frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a} \leq m \leq 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a} \text{ and } M(m,\theta,\xi) = m + \frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a}.$$

Proof

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$

(1.11') defines the three price regimes implied by the model.

1) If the price regime is h_2, then from (1.8')

$$\text{h} \leq h_4(m, \theta^a, \xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b})$$

Therefore $\frac{\xi}{\xi^b} m + \frac{\xi}{\xi^b\theta^b} - \frac{1}{\theta^a} \leq 0$ and, $m\frac{\xi}{\xi^b} + \frac{\xi^a \xi}{\xi\xi^b}\frac{1}{\theta^b} - \frac{1}{\theta^a} \leq 0$

as $\frac{\xi^a}{\xi} \prec 1$ and θ^b is by definition positive.

Therefore $m \le \frac{\xi^b}{\xi\theta^a} - \frac{\xi^a}{\xi\theta^b}$. Moreover, using (1.11'),

$$M(m,\theta,\xi) = Max\{0, m + h(\xi^a - \frac{1}{\theta^a h})\} =$$

$$Max\{0, m + \frac{\xi^a}{\xi^b}(m + \frac{1}{\theta^b}) - \frac{1}{\theta^a}\} = Max\{0, m\frac{\xi}{\xi^b} + \frac{\xi^a \xi}{\xi\xi^b}\frac{1}{\theta^b} - \frac{1}{\theta^a}\} = 0$$

2) If the price regime is h_3, the proof is identical and the results can be deducted by changing a into b and 1 into 1-m.

3) If the price regime is h_4,

$$h_4(m,\theta^a,\xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) \le h_2(m,\theta^b,\xi^{b)} = \frac{1}{\xi^b}(m + \frac{1}{\theta^b}) \text{ and}$$

$$h_4(m,\theta^a,\xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) \le h_3(m,\theta^a,\xi^{a)} = \frac{1}{\xi^a}(1 - m + \frac{1}{\theta^a})$$

Therefore $\frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a} \le m \le 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a}$ and

$$M(m,\theta,\xi) = Max\{0, m + h(\xi^a - \frac{1}{\theta^a h})\} = M(m,\theta,\xi) = Max\{0, m + h\xi^a - \frac{1}{\theta^a})\} =$$

$$Max\{0, m + \frac{\xi^a}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) - \frac{1}{\theta^a})\} = m - \frac{\xi^b}{\xi}\frac{1}{\theta^a} + \frac{\xi^a}{\xi\theta^b} \text{ Q.E.D}$$

Theorem 2

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$ Any solution to (1.13') for which prices are well defined is such that:

-at least for one s in S, prices are determined by h_4 or h_3 for m=0.

-at least for one s in S, prices are determined by h_4 or h_2 for m=1..

Proof

Suppose prices are, for $m = 0$, defined under h_2 for all s in S.

Then by Proposition A.1, $m' = M(0, \dot{\theta}(s\ ,0), \dot{\xi}(s\ ))$ is equal to 0 for all s in S.

$$h_2(0,\theta^b(s,0),\xi^b(s)) = \frac{1}{\xi^b}(m+\frac{1}{\theta^b}) = \frac{1}{\xi^b(s)}.\frac{1}{\theta^b(s,0)} = h_2(m',\theta^b(s,m'),\xi^b(s))$$

Hence, applying Lemma 1', $\theta^a \leq \frac{1}{h}U_a'(\frac{m'}{h}+\xi^a)$ and $\theta^b \geq \frac{1}{h}U'_b(\frac{1-m'}{h}+\xi_b)$ .Hence

$$\Phi(\theta)(s.0) = \begin{bmatrix} \beta\int_S \frac{1}{h'}U_a'(\xi^a(s')) & \pi(s,ds') \\ \beta\int_S \theta^b(s',0') & \pi(s,ds') \end{bmatrix} = \begin{bmatrix} \beta\int_S \frac{\xi^b(s')\theta^b(s',0)}{\xi^a(s')} & \pi(s,ds') \\ \beta\int_S \theta^b(s',0') & \pi(s,ds') \end{bmatrix}$$

As Lagrangians are always positive or nil, $\theta^b(s,0)$ as well as $\theta^a(s,0)$ can only be nil for all s in S. This makes prices at m=0 undefined. Consequently, there must be some s in S such that h is determined under h_4 or h_3.

Symmetrically, if h is under h3 and m' is equal to 1,

$$h_3(m,\theta^a,\xi^a) = \frac{1}{\xi^a}(1-m+\frac{1}{\theta^a}) = \frac{1}{\xi^a}.\frac{1}{\theta^a}$$

Hence,

$$\Phi(\theta)(s.1) = \begin{bmatrix} \beta\int \theta^a(s',1) \quad \pi(s,ds') \\ \beta\int_S \xi^a(s')\theta^a(s',1) \, {}^1\!/_{\xi^b(s')} \, \pi(s,ds') \end{bmatrix}$$

and conclusions are identical.

If m' is equal to 0 (respectively 1) while h under h4, according to Proposition A.1 $m - \frac{\xi^b}{\xi}\frac{1}{\theta^a} + \frac{\xi^a}{\xi\theta^b}$ =0 which implies that h is also under h2 (respectively h3) and hence prices are again undefined. . Consequently, there must be some s in S such that h is determined under h_4 or h_2. Q.E.D

<u>Proposition A.2</u>

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$ .Suppose that $\theta(s,m)$ is a Variant 2 solution to (1.13) such that cash constraints are always binding for one of the agents at least.

Then, for any given s in S,

either $h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$ for all m in [0,1],

or $h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s)$ for all m in [0,1].

Proof

Suppose that cash constraints are always binding for one of the agents at least. Then h is always under h2 or h3 and not under h4. Then, suppose that, for a given s in S, at least one point m_0 in]0,1[exists such that in the left (respectively right) neighborhood of m_0:

$$h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$$

while in the right (respectively left) neighborhood of a range m_0,

$$h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s).$$

Then, by Proposition A.1, and in view of the continuity in m of $\theta(s,m)$ and the fact that the functions $h(.,.,.)$ can only take one value at a time,

$$h(m,\theta(s,m),\xi(s)) = \frac{1}{\xi^b(s)}(m+\frac{1}{\theta^b(s,m)}) = \frac{1}{\xi^a(s)}(1-m+\frac{1}{\theta^a(s,m)}) \text{ for } m_0 \text{ in } [0,1]$$

. However, applying Proposition A.1 again, $M(m_0,\theta(s,m_0),\xi(s)) = 0$ and 1 which is a contradiction.

Hence, for a given s in S,

Either $h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$ for all m in [0,1],

Or $h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s)$ for all m in [0,1]. Q.E.D.

Proposition A.3

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$. Also, let us define $z_1 = \frac{\xi^a(1)}{\xi^b(1)}$ as well as $z_2 = \frac{\xi^b(2)}{\xi^a(2)}$ .and $k = \frac{0.5\beta}{1-0.5\beta}$.

If a Variant 2 solution to (1.13) exists and is such cash constraints are always binding for one of the two agents, such solution takes the following values:

$$\theta^b(1,m) = \frac{k^2 - z_1 z_2}{k(1+z_2)+(1+z_1)z_2} \;.\; \theta^a(2,m)) = \frac{k^2 - z_1 z_2}{k(1+z_1)+(1+z_2)z_1}$$

$$\theta^a(1,m) = 0.5\beta(k^2 - z_1 z_2)(\frac{1}{k(1+z_1)+(1+z_2)z_1} + \frac{1}{k(1+z_2)z_1 + (1+z_1)z_2 z_1}) \text{ and}$$

$$\theta^b(2,m) = 0.5\beta(k^2 - z_1 z_2)(\frac{1}{k(1+z_2)+(1+z_1)z_2} + \frac{1}{k(1+z_1)z_2 + (1+z_2)z_1 z_2})$$

Moreover, the two conditions below are simultaneously satisfied:

$$\frac{1}{\theta^a(1)}\frac{\xi^b(1)}{\xi(1)} - \frac{\xi^a(1)}{\xi(1)}\frac{1}{\theta^b(1)} \geq 1 \text{ and } \frac{\xi^a(2)}{\xi(2)}\frac{1}{\theta^b(2)} - \frac{1}{\theta^a(2)}\frac{\xi^b(2)}{\xi(2)} \geq 1$$

Proof

Let us apply Theorem 2 to the case of only two states in S. By assumption h is under h_4. Hence, there must be s_1 in S such that h is under h_2 for m=1 and s_2 in S such that h is under h_3 for m=0. Moreover, applying Proposition A.2, s_1 cannot be equal to s_2. Hence there must exist:

s_1 in S such that $h(m,\theta(s_1,m),\xi(s_1)) = h_2(m,\theta^b(s_1,m),\xi^b(s_1))$ for all m in [0,1], and

s_2 in S such that $h(m,\theta(s_2,m),\xi(s_2)) = h_3(m,\theta^a(s_2,m),\xi^a(s_2))$ for all m in [0,1].

Moreover, $M(m,\theta(s_1,m),\xi(s_1)) = 0$ and $M(m,\theta(s_2,m),\xi(s_2)) = 1$.

Let then calculate $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)}$.

If $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)} \leq 1$, let us identify s_1 with 1 and s_2 with 2. If $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)} \succ 1$,

let us call s_1 as 2 and s_2 as 1 so that in the end, $\frac{\xi^a(s_1)}{\xi^b(s_1)} / \frac{\xi^a(s_2)}{\xi^b(s_2)} \leq 1$,

Let us now calculate $\theta(s,m)$, the Variant 2 solution to (1.13) as

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'}U_a'(\frac{m'}{h'})\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'}U_b'(\frac{1-m'}{h'})\} \quad \pi(s,ds') \end{bmatrix}$$

Applying Lemma 1', for s'=1, $h(m,\theta(s_1,m),\xi(s_1)) = h_2(m,\theta^b(s_1,m),\xi^b(s_1))$ implies

$$\theta^a(1,m) \le \frac{1}{h} U_a'(\frac{m}{h} + \xi^a(1)) \text{ and } \theta^b(1,m) \ge \frac{1}{h} U'_b(\frac{1-m}{h} + \xi_b(1)) \text{ for all m in [0,1]}$$

For s'=2, $h(m,\theta(s_2,m),\xi(s_2)) = h_3(m,\theta^a(s_2,m),\xi^a(s_2))$ implies

$$\theta^a(2,m) \ge \frac{1}{h} U_a'(\frac{m}{h} + \xi^a(2)) \text{ and } \theta^b(2,m) \le \frac{1}{h} U'_b(\frac{1-m}{h} + \xi_b(2)) \text{ for all m in [0,1].}$$

Moreover, $M(m,\theta(1,m),\xi(1)) = 0$ and $M(m,\theta(2,m),\xi(2)) = 1$. Hence,

$$\Phi(\theta)(1,m) = \begin{bmatrix} 0.5\beta\{\theta^a(2,0) + \dfrac{1}{h_2(0,\theta^b(1,0),\xi^b(1))} U_a'(\xi^a(1))) \\ 0.5\beta(\theta^b(1,0) + \dfrac{1}{h_3(0,\theta^a(2,0)\xi^a(2)} U_b'(\dfrac{1}{h_3(0,\theta^a(2,0)\xi^a(2))} + \xi^b(2))) \end{bmatrix}$$

$$\Phi(\theta)(2,m) = \begin{bmatrix} 0.5\beta\{\theta^a(2,1) + \dfrac{1}{h_2(1,\theta^b(1,,1\},\xi(1)} U_a'(\dfrac{1}{h_2(1,\theta^b(1,1),\xi(1))} + \xi^a(1))) \\ 0.5\beta(\theta^b(1,1) + \dfrac{1}{h_3(1,\theta^a(2,1),\xi^a(2))} U_b'(\xi^b(2))) \end{bmatrix}$$

<u>with</u>

$$h_2(0,\theta^b(1,0),\xi^b(1) = \frac{1}{\xi^b(1)}(\frac{1}{\theta^b(1,0)})$$

$$h_3(0,\theta^a(2,0),\xi^a(2) = \frac{1}{\xi^a(2)}(1 + \frac{1}{\theta^a(2,0)})$$

$$h_3(1,\theta^a(2,1),\xi^a(2) = \frac{1}{\theta^a(2,1)\xi^a(2)} \qquad h_2(1,\theta^b(1,1),\xi^b(1) = \frac{1}{\xi^b(1)}(1 + \frac{1}{\theta^b(1,1)})$$

Hence, replacing and simplifying,

$$\theta(1.m) = \begin{bmatrix} 0.5\beta(\theta^a(2,0) + \dfrac{\xi^b(1)}{\xi^a(1)}\theta^b(1,0)) \\ 0.5\beta(\theta^b(1,0) + \dfrac{1}{1 + \dfrac{\xi^b(2)}{\xi^a(2)} \dfrac{\theta^a(2,1)+1}{\theta^a(2,1)}}) \end{bmatrix}$$

$$\theta(2.m)=\begin{bmatrix} 0.5\beta(\theta^a(2,1)+\dfrac{1}{1+\dfrac{\xi^a(1)(1+\theta^b(1,1))}{\xi^b(1)\theta^b(1,1)}}) \\ 0.5\beta(\theta^b(1,1)+\dfrac{\theta^a(2,1)\xi^a(2)}{\xi^b(2)}) \end{bmatrix}$$

Consequently, $\theta(s,m)=\theta(s)$ as θ does not depend on m. Hence,

$\theta^a(1)=0.5\beta(\theta^a(2)+\dfrac{\xi^b(1)}{\xi^a(1)}\theta^b(1))$ while $\theta^b(2)=0.5\beta(\theta^b(1)+\dfrac{\theta^a(2)\xi^a(2)}{\xi^b(2)})$.

Moreover,

$\theta^b(1)=0.5\beta(\theta^b(1)+\dfrac{1}{1+\dfrac{\xi^b(2)}{\xi^a(2)}\dfrac{\theta^a(2)+1}{\theta^a(2)}})$ and $\theta^a(2)=0.5\beta(\theta^a(2)+\dfrac{1}{1+\dfrac{\xi^a(1)(1+\theta^b(1)}{\xi^b(1)\theta^b(1)}})$

Hence, defining k as $k=\dfrac{0.5\beta}{1-0.5\beta}$ and $z_1=\dfrac{\xi^a(1)}{\xi^b(1)}$ as well as $z_2=\dfrac{\xi^b(2)}{\xi^a(2)}$.

$\theta^a(2)=\dfrac{k}{1+z_1+z_1\dfrac{1}{\theta^b(1)}}$ and $\theta^b(1)=\dfrac{k}{1+z_2+z_2\dfrac{1}{\theta^a(2)}}$ which implies that

$\theta^a(2)=\dfrac{k}{1+z_1+\dfrac{(1+z_2)z_1}{k}+\dfrac{z_1z_2}{k}\dfrac{1}{\theta^a(2)}})$ and $\theta^a(2)=\dfrac{k^2-z_1z_2}{k(1+z_1)+(1+z_2)z_1}$

Similarly, $\theta^b(1)=\dfrac{k^2-z_1z_2}{k(1+z_2)+(1+z_1)z_2}$. Consequently

$$\theta^a(1)=0.5\beta(k^2-z_1z_2)(\frac{1}{k(1+z_1)+(1+z_2)z_1}+\frac{1}{k(1+z_2)z_1+(1+z_1)z_2z_1})\text{ and}$$

$$\theta^b(2)=0.5\beta(k^2-z_1z_2)(\frac{1}{k(1+z_2)+(1+z_1)z_2}+\frac{1}{k(1+z_1)z_2+(1+z_2)z_1z_2}).$$

For this to be a solution, it is first necessary for θ to be positive or nil, i.e

that $k^2\geq z_1z_2$ which implies: $\dfrac{0.5\beta}{1-0.5\beta}=k\ \geq\sqrt{z_1z_2}$ or $\beta\geq\dfrac{2\sqrt{\dfrac{\xi^a(1)}{\xi^b(1)}/\dfrac{\xi^a(2)}{\xi^b(2)}}}{1+\sqrt{\dfrac{\xi^a(1)}{\xi^b(1)}/\dfrac{\xi^a(2)}{\xi^b(2)}}})$.

Moreover, it has to be ensured that prices are never defined under h_4, i.e that:

-for s= 1 and all m in [0,1], $\dfrac{1}{\xi(1)}(\dfrac{1}{\theta^a(1)}+\dfrac{1}{\theta^b(1)})\geq\dfrac{1}{\xi^b(1)}(m+\dfrac{1}{\theta^b(1)})$

-for s=2 and all m in [0,1], $\frac{1}{\xi(2)}(\frac{1}{\theta^a(2)}+\frac{1}{\theta^b(2)}) \geq \frac{1}{\xi^a(2)}(1-m+\frac{1}{\theta^a(2)})$

which implies:

1) $\frac{\xi^b(1)\theta^b(1)}{\xi^a(1)+\xi(1)\theta^b(1)} \geq \theta^a(1)$ and 2) $\frac{\theta^a(2)}{\theta^b(2)}) - (1+\frac{\xi^b(2)}{\xi^a(2)})\theta^a(2) \geq \frac{\xi^b(2)}{\xi^a(2)}$ .Q.E.D.

Theorem 3

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5\ 0.5 \\ 0.5\ 0.5 \end{bmatrix}$. Suppose finally that the endowment matrix $\xi(.)$ is such that $\frac{\xi^a(1)}{\xi^b(1)} = \frac{\xi^b(2)}{\xi^a(2)}$=z.

Then a Variant 2 solution to (1.13) exists if

$\frac{2z}{1+z} \leq \beta < \beta_{Max} = \frac{-2z+\sqrt{(20z^2+16z)}}{2(1+z)}$ and is such that cash constraints are always binding for one of the two agents.

<u>Proof</u>

Applying Proposition A.3, $\theta^b(1,m) = \frac{k-z}{(1+z)}.\ = \theta^a(2,m))$

$$\theta^a(1,m) = 0.5\beta(k^2-z^2)(\frac{1}{(1+z)(k+z)}+\frac{1}{(1+z)(k+z)z})$$

$$= 0.5\beta(k^2-z^2)(\frac{1+z}{(1+z)(k+z)z}) = \frac{k(k-z)}{(k+1)z} = \theta^b(2,m).$$

Let us now check the two conditions put forward in Proposition A.3, $\frac{\xi^b(1)\theta^b(1)}{\xi^a(1)+\xi(1)\theta^b(1)} \geq \theta^a(1)$ can be written $\frac{z}{(1+z)} \geq k^2/(k+1) = \frac{\beta^2}{4-2\beta}$. This implies $(1+z)\beta^2 + 2\beta z - 4z \leq 0$ Roots of this polynomial are both positive and negative.

$$\text{Then } \frac{2z}{1+z} \leq \beta < \beta_{Max} = \frac{-2z+\sqrt{20z^2}+16z)}{2(1+z)}$$

$\frac{\theta^a(2)}{\theta^b(2)}) - (1+\frac{\xi^b(2)}{\xi^a(2)})\theta^a(2) \geq \frac{\xi^b(2)}{\xi^a(2)}$ can be written $\frac{\theta^a(2)}{\theta^b(2)}) - (1+z)\theta^a(2) \geq z$, leading to the same condition. Q.E.D.

References

Acharya, V.V., "A theory of systemic risk and design of prudential bank regulation." Journal of Financial Stability 5 (2009)

Aglietta, M and Moutot, P "Le risque de système et sa prévention", Cahiers économiques et monétaires, Banque de France, n° 41, (1993)

Algan, Y. et Ragot X., « Monetary policy with Heterogeneous Agents and Borrowing constraints", *Review of Economic Dynamics*., 2010

Ashley, R. and Patterson, D. "Linear versus Nonlinear Macroeconomics: a Statistical Test." International Economic Review, 30 n° 3 (August 1989): 685-704.

Beinhocker, E.D. "The origin of wealth", Harvard Business School Press, 2006

Bewley, T., 1980, "The Optimum Quantity of Money," in Models of Monetary Economics, ed.by John Kareken and Neil Wallace. Minneapolis, Minnesota: Federal Reserve Bank, 1980.

Bouchaud, J.-P The (unfortunate) complexity of the economy, Presentation at the 2010 ECB conference, http://www.centre-dalembert.u-psud.fr/documents/pdf2010-2011/jpbouchaud.pdf

Bouchaud J.-P, Gefen Y., Potters, M.and Wyart, M.. Fluctuations and response in financial markets: The subtle nature of "random" price changes. Quantitative Finance, 4(2):176–190, 2004;

Bouchaud J.-P, J. Kockelkoren, and M. Potters. Random walks, liquidity molasses and critical response in financial markets. Quantitative Finance, 6(2):115–123, 2006.

Burkhard, H. and Mauβner A., “Dynamic General Equilibrium Modelling”, November 4, 2004, Springer-Verlag

Clower, R. W. “A Reconsideration of the Microfoundations of Monetary Theory” Western Economic Journal, 6 n° 1 (December 1967) 1-9.

Clower, R.W. and Howitt P. “The Transactions Theory of the Demand for Money” Journal of Political Economy, 86 n° 3, (June 1978) 449-466

Cole, H. “Financial Structures and International Trade.” International Economic Review, 29 n°2 (1988): 237-59.

Coleman, W.J. “Money, Interest and Capital.” Ph.D. dissertation, University of Chicago, 1986.

Coleman, W.J. “Money and Capital in a Cash In Advance Economy.”, Working Paper , Board of Governors of the Federal Reserve System, September 1990.

Conze, A. “Problèmes d´équilibre dynamique sous contraintes d’endettement. Problèmes d´évaluation dáctifs contingents.” Ph.D. dissertation, Université Paris IX Dauphine, 1990.

Cutler, D. M., Poterba, J. M. and Summers, Lawrence H., “”What Moves Stock Prices?” (July 1989). NBER Working Paper Series, Vol. w2538, 1989.

De Boissieu, C. “Les vitessses de circulation de la monnaie. Une approche conflictuelle”, Paris: Editions Cujas, 1975.

Duffie, D., Geanakoplos J., Mas/Colell A., and McLennan, A. "Stationary Markov Equilibria." Research Paper Series, n° 1079, Graduate School of Business-Stanford University, July 1989.

Farmer D. and Geanakoplos J. "The virtues and vices of equilibrium and the future of financial economics'" , arXiv:0803.2996v1 [q-fin.GN] 20 Mar 2008

Freeman, `S. "The Payments System, Liquidity, and Rediscounting," *American Economic Review*, December 1996.

Feenstra, R.C. "Functional Equivalence between Liquidity Costs and the Utility of Money." Journal of Monetary Economics, 17 (1986): 271-291.

Fuerst, T. "Liquidity, loanable funds, and real activity". Journal of Monetary Economics, 29(1992): 3-24.N

Fuhrer, Jeffrey C., "Habit Formation in Consumption and Its Implications for Monetary Policy Models", Federal Reserve Bank of Boston , April, 2000

Grandmont, J.M. "On Endogenous Competitive Business Cycles." Econometrica, 53 (1985): 995-1045.

Guidotti, E.P. "Exchange Rate Determination, Interest Rates, and an Integrative Approach for the Demand for Money." Journal of International Money and Finance, 8 (1989): 29-45.

Hansen, L.P. and Singleton, K.J. "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns." Journal of Political Economy, 91 (1983): 249-265.

Hansen,G.D.and Imrohoroglu, A. "The role of Unemployment Insurance in an Economy with Liquidity Constraints and Moral Hazard" Journal of Political Economy. 100 (1992) 118-142.

Helpman, E. "An Exploration in the Theory of Exchange Rate Regimes." Journal of Political Economy, 89 (1981): 865-890

Helpman, E. and Razin, A. "Towards a Consistent Comparison of Alternative Exchange Rate Regimes." Canadian Journal of Economics, 12 (1979): 394-409.

Helpman, E. and Razin, A. "A Comparison of Exchange Rate Regimes in the Presence of Imperfect Capital Markets." International Economic Review, 23 (1982):365-388.

Helpman, E. and Razin, A. "Dynamics of a Floating Exchange Rate Regime." Journal of Political Economy, 90 (1982): 728-754.

Hodrick, R. "Risk, Uncertainty, and Exchange Rates." Journal of Monetary Economics, 23 (1989): 453-459.

Hodrick, R., N., Kocherlakota, and D. Lucas, 1991, The variability of velocity in cash-in-advance models, Journal of Political Economy 99(2), 358-384.

Huang, R.D. "The Monetary Approach to Exchange Rate in an Efficient Foreign Exchange Market. Test based on Volatility." The Journal of Finance, 36 (1981): 421-435.

Imrohoroglu, A. "The welfare const of inflation under imperfect insurance", Journal of Economic Dynamics and Control,16(1992) 79-91

Kocherlakota, Naranaye R,,"Money:What's the Question and Why Should We Care About the Answer? American Economic Review, Vol 92, N2, May 2002, pp.58-61.

Kiyotaki N.and Wright R., "On Money as a Medium of Exchange", The Journal of Political Economy, Vol97, N4, August 1989, pp927-954

Kiyotaki N.and Wright R., "A Search-Theoretic Approach to Monetary Economics", American Economic Review, Vol83, N1, March 1993, pp63-77

Kollman, R. “World Business Cycles and Incomplete International Asset Markets. Ph.D. dissertation, University of Chicago, November 1990.

Laidler, D, “Lucas, Keynes, and The Crisis”, Journal of the History of Economi Thought (2010), 32 : 39-62 Cambridge University Press

Lucas, R.E. “Interest Rates and Currency Prices in Two Country World.” Journal c Monetary Economics, 10 (1982): 335-360.

Lucas, R E., Jr. 1980. Methods and problems in business cycle theory. Journal c Money, Credit, and Banking 12 (November, part 2): 696–715.

Lucas, R.E. and Stockey, N. “Money and Interest in A Cash in Advance Economy’ Econometrica, 55 (1987): 491-513.

Manuelli, R. and Sargent, T.J. “Longer Trading Periods in the Townsend Turnpik Model.” Working Paper, University of Stanford, October 1988.

Medio, A.”Invariant Probability Distributions in Economic Models”, i Macreconomic Dynamics ,8,2004, 162-187.

Meese, R.A. and Singleton, K.J. “Rational Expectations and the Volatility of Floatin Exchange Rates.”, International Economic Review, 24 (1983): 722-733.

Moutot, P. “Preliminary Steps Toward a Solution of Non-Linear Dynamic an Stochastic Two-Agent Models.” Manuscript February 1987.

Moutot Philippe P., “Money in a Non-linear Dynamic and Stochastic Two-Agen Model,” PhD dissertation, University of Chicago, August 1991.

Popper, K., Logik der Forschung, 1934

Rajan, Raghuram G, 2010, “Fault lines”, Princeton University Press, Princeton, N.J.

Scheinkman, J.A. “Nonlinearities in Economic Dynamics”, The Economic Journal, 100 (Conference 1990): 33-48.

Scheinkman, J.A. and Weiss, L. “Borrowing Constraints and Aggregate Economic Activity.” Econometrica, 54 (1986): 23-45

Schiller, R.J. “Do Stock Prices Move too Much to Be Justified by Subsequent Change in Dividends.” American Economic Review, 71 (1981): 421-435.

Schiller, R.J. (1991): Market Volatility, Cambridge, MIT Press

Stockman, A. “A Theory of Exchange Rate Determination.” Journal of Political Economy, 88 (1980): 673-697.

Stockman, A. and Dellas, H. “International Portfolio Diversification and Exchange Rate Variability.” Journal of Monetary Economics, 26 (1989): 271-289.

Townsend, R.M. “Models of Money with Spatially Separated Agents.” In J.H. Kareken and N. Wallace, eds, Models of Monetary Economics, Minneapolis: Federal Reserve Bank of Minneapolis, 1980.

Wyart, M. Bouchaud, J.-P. Kockelkoren, J. Potters, M.and. Vettorazzo, M. “Relation between bid-ask spread, impact and volatility in double auction markets”. Quantitative Finance, 8(1):41-57, 2008

Wen, Y. “Lucas meets Baumol and Tobin.” Working Paper 2010-014B, Federal Reserve of St Louis

Woodford, M. “Stationary Sunspot Equilibria in a Financed Constrained Economy.” Journal of Economic Theory, 40 (1986): 128-137.

Woodford, M., 2003, “Interest and Prices”, Princeton University Press, Princeton N.J.