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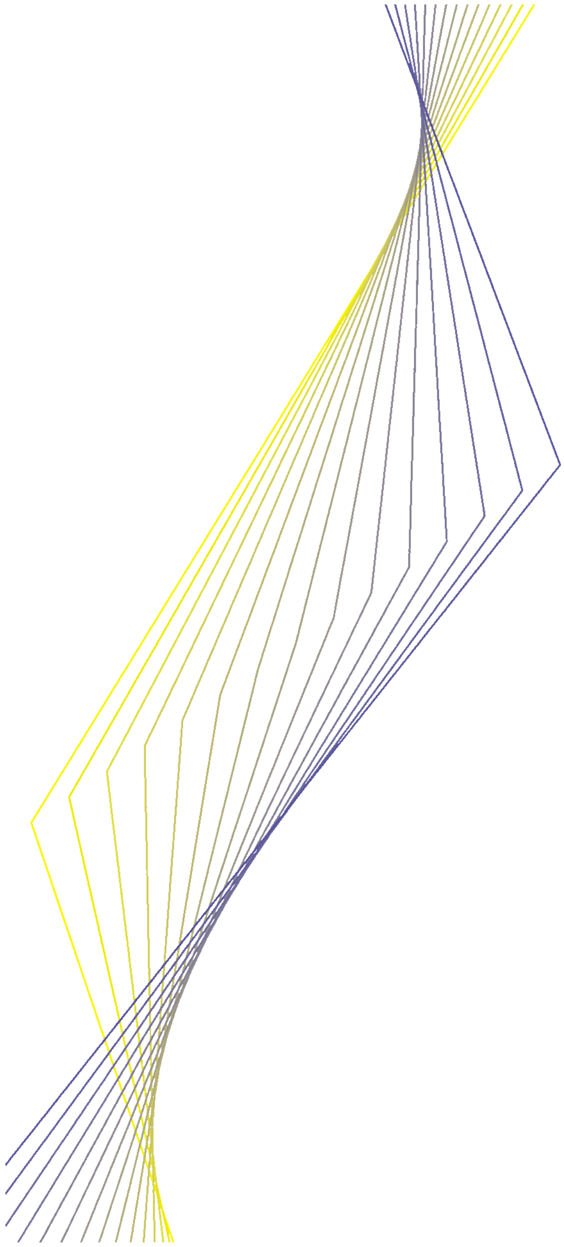

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THE FUNCTIONAL FORM OF YIELD CURVES

BY VINCENT BROUSSEAU*

May 2002

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Contents

Abstract

Yield curves built from liquid instruments tend to exhibit specific features, both in term of smoothness and in term of patterns. The paper presents empirical evidence that those liquid yield curves frequently conform to a specific functional form. This specific functional form is predicted by a particular arbitrage pricing model. The paper also examines the possible interpretations of this phenomenon.

JEL classification codes: G10; G12; G13

Keywords: Yield curves, affine models, bootstrapping, swaps, bonds, futures, functional form

Executive Summary

For many years, the shape of the US interest rates term structure has been consistently regular and smooth. The same has been observed in several European currencies from the mid-nineties onwards. This has led market practitioners and academic researchers to question to what extent interest rate term structures conformed to a specific shape. In that event, a second question was whether it was possible to identify this shape, to recognise the pattern displayed by term structures. The author could observe the efforts conducted around 1993-1994 by market practitioners in the French franc market to fit the yield curve with a fairly simple, albeit not trivial, mathematical form. Similar efforts have been undertaken inter alia by Nelson and Siegel or by Svensson.

Interest rate term structures, or yield curves, are built from a potentially very large number of prices (or of yields) of debt instruments. Identifying the specific pattern displayed by these curves would allow capturing their information content into a much small number of parameters. This would facilitate their use by practitioners (e.g. market makers, proprietary traders or institutional asset managers) for a variety of purposes, ranging from asset pricing to interest rate risk management. This would also be beneficial to financial economists, who could determine the shape of the yield curve and summarise its informational content into a few numbers relatively easy to read and interpret.

This paper proposes a method to identify such a recurrent shape in American and European yield curves through a form of pattern recognition. It also provides evidence regarding the effective recurrence of specific patterns in the shape of yield curves. In essence, we scan the term structures observed at various dates, for various countries, and for various markets (either bonds or money markets) in order to identify whether a particular structure is frequently present and identifiable. In more technical words, we examine whether the term structures frequently match a given functional form.

The application of the concept and methodology of the famous Black and Scholes model to interest rates term structures, translates into another famous model, or rather class of models, known as the Heath-Jarrow-Morton model (hereafter HJM model). The HJM model is compatible with any possible term structure in the sense that it does not prescribe any particular functional form. Duffie and Kan (1996) devised a particular case of the HJM model, which we will use in its simplest version, known as the Duffie and Kan one-factor model (hereafter DK1 model): Contrary to the HJM model, the DK1 model does constrain the functional form of the term structure.

After scanning some 17,000 observations of term structures, we conclude that those term structures commonly conform to the functional form prescribed by the DK1 model. They match that functional form not only frequently, but also precisely. A particularly noteworthy finding is that, while a theoretical DK1 yield curve is entirely characterised by as few as 5 parameters, actual observed yield curves composed of up to 160 individual instrument prices fit that theoretical pattern with a precision of only a few basis points.

This positive result could be credited to the accuracy of the DK1 model. However, another possible explanation relates to the overwhelming importance of quantitative trading. This leads to a possible interpretation of the accuracy with which an empirical yield curve fits the DK1 theoretical curve as a measure first of the liquidity, and then of the benchmark status of that empirical curve. In other terms, the more a curve is used a pricing reference for the whole market, the more it would be likely to conform to a theoretical patters used by market practitioners for risk management purposes. The improving fit of the observed and theoretical curves over the past years, in Europe especially, may be interpreted as providing support to this explanation.

1 INTRODUCTION

Yield curves built from liquid instruments behave in a particular manner. The *motion* of those curves, as well as the *form* that they display at a given point of time, exhibit recognisable features. This suggests that the shape of liquid yield curves obey particular laws, constraining their shape as well as their dynamics. In well-developed financial markets, such as those of the US or the euro area, those laws play a crucial role, alongside the monetary policy stance and expectations, in the formation of interest rates across the maturity spectrum.

Understanding the formation of interest rates is crucial for risk management, for trading activity and for monetary policy. Indeed, monetary policy is, in major economies, implemented in financial markets, which also form an essential part of its transmission process. As a consequence, empirical regularities either in the statics or in the dynamics of yield curves should be identified, studied, and if possible explained.

This paper focuses on a static or cross-sectional dimension, i.e. it examines the pattern displayed by yield curves at a given instant, from a given snapshot. The analysis covers the core interest rate markets in the United States and the euro area. Yield curves are built using daily data from the money market as well as the markets for government bonds.

The following findings are evidenced: very frequently, the yield curves contained in our (extended) samples fit a particular functional form. In addition, very frequently, the accuracy of the fit between the theoretical form and the empirical curve is remarkably high. While the theoretical curve is entirely determined by only five parameters, and the empirical curve is made of much larger number of individual instrument prices, the distance between the two curves remains within a few basis points. Graphically, the two curves are most often difficult to distinguish.

The functional form, which captures frequently - but not always - the shape of the yield curves is itself derived from a specific model. It would then be tempting to conclude that the model is a correct representation of the reality. Such a hasty interpretation may however be subject to caution. The model predicts the form of the curve with a high degree of success, but it also predicts other things, which are plainly false. For example, did the model hold, then its parameters should imperatively stay constant over time (because the model is logically grounded on that assumption). In practice, they do not.

The quality of the fit between the theoretical curves and the curves empirically observed is however too good to be explained by mere coincidence. If it is not the consequence of the accuracy of the underlying model as a description of reality, then another explanation is required. One possible such explanation can be proposed on the basis of a qualitative description of the behaviour of market practitioners (traders). Confirming (or rejecting) this interpretation by more formal methods would now require the analysis of very large and complex databases of tick data as well as by extremely sophisticated algorithms.

The paper is organised as follows. Section 2 distinguishes between the functional forms that one might try to fit to empirical curves. It introduces the properties of parsimony, viability and completeness of those forms. The particular form that we are going to fit to the empirical data possesses those three qualities. Section 3 deals in some details with the technical aspects of data and computations. Section 4 presents and discusses the results, and then suggests an interpretation of those results. Section 5 concludes.

2 PARSIMONY, VIABILITY, COMPLETENESS

This paper aims at finding evidence of an empirical phenomenon. In short, we want to prove that (liquid) yield curves tend to exhibit a specific functional form or pattern. We will bring that evidence in sections 3 and 4 below. The positive identification of such a functional form is always, in itself, an interesting finding, but the interpretation of this finding, its practical use, (and the questions it may raise,) depend on some properties of the functional form. In this section, we will define three properties of those forms: Parsimony, viability and completeness. We will give examples of functional forms having only parsimony, and of functional forms having the three properties. We will moreover explain what bearing those properties have on the practical use and on the interpretation of a positive finding.

2.1 The yield curve

This paragraph encompasses some concepts and results, regarding the yield curve as well as its arbitrage-pricing model. Most of them are well known. Their presentation here serves mainly the purpose of clarifying some definition and notation issues.

2.1.1 The yield curve: Definitions and notations

Yield curves may be formally defined as functions of a continuous time parameter, which associates an interest rate to a maturity within a (theoretically unbounded) set.

Yield curves may be expressed in two ways. These two ways are equivalent and hence contain the same quantity of information: *zero coupon interest rate* curves and *instantaneous forward interest rate* curves. Let us denote with z(t) the zero coupon interest rate at maturity t and with f(t) the forward rate interest rate at maturity t (both interest rates being continuously compounded). Let us denote with P(t) the spot price of the zero coupon of maturity t, i.e. the present value of one currency unit to be paid at t. One then has:

$$P(t) = \exp(-t\, z(t)) \tag{1}$$

and:

$$\frac{P'(t)}{P(t)} = -f(t) \tag{2}$$

Hence one has the relationship:

$$f(t) = z(t) + t\,\frac{dz(t)}{dt} \tag{3}$$

This change of variables is duly inversible, as one has:

$$z(t) = \frac{1}{t}\int_0^t f(s)\,ds \tag{4}$$

Conversely, formula (3) can be obtained from formula (4).

2.1.2 Arbitrage pricing models: General overview

A model is said to be *viable* if it does not allow for arbitrage opportunities. This is a sort of minimal requirement for a model. If this requirement is not fulfilled, then prices become indeterminate. Indeed, the mere existence of a single arbitrage effectively "poisons" the pricing of any financial instrument: The price becomes an indeterminate form.

A model is said to be *complete* if it allows for the synthetic replication, in the sense of Black and Scholes, of any possible contingent claim. The key example of a viable model that is moreover complete is obviously Black and Scholes (1973), for the very concept of pricing by synthetic replication originates in this paper. If the requirement of being complete is not met, then the standard Black and Scholes definition of an option fair price no longer apply. Thus one is left without a definition of option prices that would economically make sense (and that would be logically consistent).

Even if a model is viable, it is by no means certain that it is also complete. It is standard and well known that viability and completeness depend on properties of the process of (relevantly discounted) prices of the financial instruments (that are encompassed in the model). The model is viable if this process is equivalent to a martingale, it is moreover complete if this martingale is unique – here we do not need to elaborate on those matters [1].

2.1.3 Arbitrage pricing models of the yield curve

In the case of the yield curve, the minimal task of an arbitrage pricing model is to provide price at least for zero-coupon bonds. Those zero-coupon bonds price always take the form:

$$P(t) = \mathbf{E}_u \left[e^{-\int_u^t r_s \, ds} \right] \tag{5}$$

where u is the current date, r_s is the short-term rate at future date s>u and **E** is the expectation knowing all up to current date u, under some measure. The (relevantly discounted) price is called the relative price, it is frequently denoted with Z(t), and it is defined as:

$$Z(t) = e^{-\int_0^t r_s \, ds} \, P(t) \tag{6}$$

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[1] Completeness is actually linked to the geometric position of this martingale in the set of all martingales (adapted to the same filtration). Completeness means that the martingale occupies, in this set, an extremal position. This topic is far beyond the scope of the present paper, and one can only refer the reader to Harrison and Pliska (1981 and 1983) for the link between completeness and extremality, to Jacod for the technicalities, and to Elliott and Kopp (1999) for an accessible and complete demonstration of the equivalence between the two thing in the case of discrete time.

where 0 is some fixed reference date chosen in the past [2]. Due to equation (5), Z(t) has dynamics

$$Z(t) = \mathbf{E}_u \left[e^{-\int_0^t r_s \, ds} \right] \tag{7}$$

By its very construction, (7) is a martingale [3]. Thus, models defined from (5) or (7) are necessarily *viable*. The question whether or not they are *complete*, nonetheless, depends on the form of the short rate process $r_{\cdot}$. To the author's knowledge, no general criterion on the process $r_{\cdot}$ has ever been established (the interested reader may consult Harrison and Pilska (1981), (1983) and the various references to Jacod that are made in those papers).

The short rate process $r_{\cdot}$ can be specified in various ways. For example, Elliot, Hunter and Jamieson (2000) make it a (continuous time, discrete state) Markov chain, however this paper does not prove completeness (nor lack thereof). By contrast, the very general specification of Heath Jarrow and Morton (1992) (hereafter HJM), making Ito processes of short rate and forward rates, possesses completeness [4].

The HJM framework is extremely general. "*The model, [...], captures the essence of all the existing pricing models*" (HJM (1992), § 5 p. 89). The authors mention that their framework encompasses Vasicek (1977), Brennan and Schwartz (1979), Langetieg (1980), Artzner and Delbaen (1988), and they take a closer look at the case of Cox-Ingersoll and Ross (1985) (hereafter CIR). Any particular case, i.e. any model nested in HJM, inherits from HJM both viability and completeness.

2.2 The functional form of the yield curve

We present now briefly the concept of functional form (of a yield curve), at the centre of the analysis throughout this paper. The current section first presents examples of models in which the yield curve is assumed (explicitly or implicitly) to match a particular functional form, with due attention to some qualities of this functional form. We will refer to those qualities as to the parsimony, the viability and the completeness of the form. The section then briefly explains why functional forms (of yield curve) might be of interest, and indicates why that interest depends on the parsimony, on the viability and of the completeness of the functional form.

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[2] Both the definition of the relative price and its notation with a Z originate in Heath, Jarrow and Morton (1992) (see § 3 p. 83).

[3] With respect to the filtration $\mathfrak{F}_u$, $0 \le u$, where $\mathfrak{F}_u$ denote appropriate σ-algebras of functions defined on [0,u]. Z(.) is a vector-valued martingale, and the Z(t) are real-valued martingales, with respect to this filtration $\mathfrak{F}_u$.

[4] This is proven in Proposition 3, § 5 p.86, equivalence between (16) and (18). Note that "contingent claim", in the HJM paper (see § 5 p. 87) refers to the definition of Harrison and Pliska (see HP, § 2.2 p. 226). This is simply a technical necessity and it does not alter the economic substance of the result.

2.2.1 Parsimonious form, viable form, complete form

In this sub-section, we will consider a few examples of analytical representations of the yield curve. An analytical representation of the yield curve is a closed-form formula giving the zero-coupon rate (or equivalently the instantaneous forward rate) as a function of the maturity. This sub-section will put a special emphasis on a particular question: Are those analytical representations of the yield curves consistent with the absence of arbitrage opportunities hypothesis?

The answer will be negative in the first case (up to a small sub-case), and positive in the second case. This second case will play an essential role in the paper, since the purpose is to examine whether or not this analytical representation of the yield curve actually matches empirical (liquid) yield curves.

A first, important remark, pertains to the number of parameters that are needed to write down the functional form. A functional form with too many parameters has neither significance nor usefulness. Of interest are only those functional forms that can be specified by a small number of parameters, as they, and only they, give a parsimonious representation of the yield curve. In what follows, we will refer to that by using the expression "parsimonious functional form".

The idea to fit an empirical yield curve with some prescribed functional form is not new. The functional form is not always supposed to derive from a model. An example at hand is the Nelson & Siegel (1987) formula, which reads:

$$f(t) := \beta_0 + \beta_1 \exp\left(-\frac{t}{\tau}\right) + \beta_2 \frac{t}{\tau} \exp\left(-\frac{t}{\tau}\right) \tag{8}$$

or equivalently:

$$z(t) := \beta_0 + \beta_1 \frac{\tau}{t}\left(1-\exp\left(-\frac{t}{\tau}\right)\right) + \beta_2 \left(\frac{\tau}{t}\left(1-\exp\left(-\frac{t}{\tau}\right)\right) - \exp\left(-\frac{t}{\tau}\right)\right) \tag{9}$$

where z denotes the zero-coupon rate and f the instantaneous forward rate. The variable, (i.e. the argument of the z and f functions,) is here the time-to-maturity t. The parameters are here the three coefficients β_i and the time scale parameter τ. This "scarcity" of the parameters strongly constrains the possible forms of the curve, which is bound to have at most one hump. To improved realism, Svensson (1994) proposes a generalized form, whose 6 parameters enable the yield curve to exhibit a second hump:

$$f(t) := \beta_0 + \beta_1 \exp\left(-\frac{t}{\tau_1}\right) + \beta_2 \frac{t}{\tau_1} \exp\left(-\frac{t}{\tau_1}\right) + \beta_3 \frac{t}{\tau_2} \exp\left(-\frac{t}{\tau_2}\right) \tag{10}$$

$$\begin{aligned} z(t) := \beta_0 &+ \beta_1 \frac{\tau_1}{t}\left(1-\exp\left(-\frac{t}{\tau_1}\right)\right) + \beta_2 \left(\frac{\tau_1}{t}\left(1-\exp\left(-\frac{t}{\tau_1}\right)\right) - \exp\left(-\frac{t}{\tau_1}\right)\right) \\ &+ \beta_3 \left(\frac{\tau_2}{t}\left(1-\exp\left(-\frac{t}{\tau_2}\right)\right) - \exp\left(-\frac{t}{\tau_2}\right)\right) \end{aligned} \tag{11}$$

It should be mentioned that those forms are particularly popular among central bankers, despite their lack of theoretical grounds. As mentioned earlier, they cannot be derived from a (viable) model, with the only exception of a highly particular specification (10-11) - where 3 out of the 6 parameters are fixed. Filipović (2000a) and Filipović (2000b) demonstrate that the specifications of Svensson (1994) and Nelson and Siegel (1987) are not consistent with the absence of arbitrage opportunities hypothesis [5]. It might be useful to detail what this exactly means.

It does not mean, of course, that the absence of arbitrage opportunities prevents an actual yield curve to meet the particular pattern of Nelson & Siegel, or Svensson, at a particular point of the time. Indeed, the rather general HJM framework, that contains only viable models, can be made consistent with virtually any initial pattern of the yield curve (up to so-called "mild" technical conditions of regularity, without economic substance) . What Filipović relevantly points out is that under (truly random) arbitrage-free motion, this pattern will be immediately lost, unless in one trivial case (the one in which the Svensson functional form encompasses the generalised Vasicek model's functional form). *"There is no possibility for modelling the* [yield curve] *by exponential-polynomial families with varying exponents driven by diffusion processes.*" (Filipović (2000a), chap. 6 § 8 p. 94).

This first example of the Nelson-Siegel form and its Svensson extension exemplifies the possibility of a negative consistency result. In other cases though, the functional form is actually consistent with the absence of arbitrage opportunities hypothesis, as it just derives from a (viable) model. To put it differently, there exist models that produce an explicit, closed-form (that is the key point) formula for the yield curves. The one we intend to focus on is the Duffie and Kan 1-factor model (hereafter DK1) [6].

In the DK1 model the short-term rate is assumed to follow the 1-dimension diffusion process:

$$dr_s = (a - b\,r_s)\,ds + \sqrt{\mu^2 + r_s\,v^2}\;dW_s \tag{12}$$

and the bond price of maturity t is obtained by formula (5). The parameters are here the four coefficients a, b, μ and ν.

The formulae giving the zero-coupon rate z(t) and the instantaneous forward rate f(t) are:

$$f(t) = \left(\frac{2\,\gamma}{(\gamma+b)\left(e^{\gamma t}-1\right)+2\,\gamma}\right)^2 e^{\gamma t} r +$$

$$2\,e^{\gamma t}\gamma\frac{\left(e^{\gamma t}(\gamma+b)^2\left(b\mu^2+av^2\right)+2(\gamma+b)\mu^2v^2+2\,av^4\right)}{\left((\gamma+b)\left(e^{\gamma t}-1\right)+2\,\gamma\right)^2 v^4} - 2\left(\frac{a}{\gamma-b}+\frac{\mu^2}{(\gamma-b)^2}\right) \tag{13}$$

or equivalently:

[5] See Filipović (2000a), chapter 6 § 8, theorems 6.14 and 6.15 p. 93-94, and § 9 p. 95.

$$z(t)=\frac{2\left(e^{\gamma t}-1\right)}{\left((\gamma+b)\left(e^{\gamma t}-1\right)+2\gamma\right)t}\; r-2\frac{b\mu^2+a v^2}{v^4\,t}Log\left(\frac{2\gamma e^{\frac{(b+\gamma)}{2}t}}{(\gamma+b)\left(e^{\gamma t}-1\right)+2\gamma}\right)+$$

$$\frac{\mu^2}{v^2}\left(\frac{2\left(e^{\gamma t}-1\right)}{\left((\gamma+b)\left(e^{\gamma t}-1\right)+2\gamma\right)t}-1\right) \qquad (14)$$

where γ stands for:

$$\gamma:=\sqrt{b^2+2v^2} \qquad (15)$$

The celebrated Vasicek model [7] and Cox-Ingersoll-Ross model (hereafter CIR) are special cases of the DK1 model. They correspond respectively to the case $\mu=0$ and to the case $v=0$ [8].

The DK1 model is itself a special case of the HJM model. To check that proposition, one has to check whether the volatility of the forward rate satisfies the no-arbitrage condition C.6 in HJM. Those authors name this condition the "*forward rate drift restriction*", which is self-explaining, and this condition is the one and only restriction that they impose to the curve's dynamic.

This no-arbitrage condition (HJM (1992), § 4 eq. (18) p. 86) is in particular satisfied if:

$$\alpha(t)=\ \sigma(t)\ \int_0^t\sigma(s)ds \qquad (16)$$

where $\alpha(t)$ is the drift, and $\sigma(t)$ the volatility, of the instantaneous forward rate f(t): This is simply specialising, in this equation (18), the term called $\varphi_i(t)$ to the particular value 0.

One uses now the fact that the forward rate f(t) is given by (13) as a function of r and t only, and r obeys the SDE (12). Thus, (16) reads indeed as follows:

$$\left(\frac{\mu^2+r\,v^2}{2}\right)\left(\int_0^t\frac{\partial f(s)}{\partial r}ds\right)^2-(a-b\,r)\left(\int_0^t\frac{\partial f(s)}{\partial r}ds\right)+\int_0^t\frac{\partial f(s)}{\partial s}ds\ =\ 0 \qquad (17)$$

Thus, the DK1 model is a special case of HJM if and only if the integro-differential equation (17) holds true. This is presented by HJM (speaking actually about the CIR sub-case) as an easy thing: "*In fact, to check condition C.6, one can easily verify that expression (18) is satisfied*" (HJM (1992), § 8 p. 97). This calculation is, though direct, somewhat tedious, and one might not endorse the HJM appreciation about its easiness. However, the (formal) verification of (17) can be computerised, as is shown in Annex.

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6 A detailed presentation of the properties of the DK1 model or of the family of affine models is beyond the scope of the present paper. One can refer the reader to Duffie and Kan (1993), Brown and Schaeffer (1994), section3 pp.566-568, and to of Frachot (1996), section IV § 4-5 pp 96-110.

7 We mean here the original one, by opposition to the generalised one.

8 The CIR model captures the essence of DK1 much better than the Vasicek model does. Indeed, it can be proved that any DK1 curve is equal to some CIR curve plus some constant (which constant

Because the DK1 model is a special case of the HJM model, it is not only viable, but moreover complete. It is consistent with the absence of arbitrage opportunities hypothesis *and* it allows for a consistent pricing, rightly defined *à la* Black and Scholes, of any contingent claim. In what follows, we will refer to that by forging the expressions "viable functional form" and "complete functional form", - notwithstanding the fact that viability and completeness are, strictly speaking, properties of the models and not of the functional forms.

To summarise, the calculus (17) proves the functional form (13-15) to be both viable and complete, while simple visual inspection suffices to prove that the functional form (13-15) is parsimonious.

2.2.2 Importance of the properties of the form

Suppose that a particular form has been found to match some empirical curve.

As we already noted, the worth of such a finding depends of the parsimony of the form. An arbitrarily complicated form could match any set of empirical data with any precision. If the form is parsimonious, it has at least two merits.

- *Practical usefulness*: A portfolio manager may represent the state of any portfolio, however complicated, as a function of the few coefficients that entirely specify the form. He or she may compute the hedge of the position by deriving its value with respect to those few coefficients.
- *Scientific interest*: The curves have been built from rather large collections of instruments prices, be it swaps, futures, money market rates or government bonds. One should a priori exclude that an empirical curve made of dozens of data actually follows a mathematical formula specified by only a small number of parameters, unless there is some rationale for it. It is worth investigating what this rationale might be.

Suppose now that the form is moreover viable. We then have something more.

- *Practical usefulness*: A portfolio manager or an arbitragist may imply the values of the coefficients. This is in essence deriving a implied short term interest rate process, by inverting formula (7). This indicates which sorts of motion of the short rates are broadly consistent with the observed curve. It may help identifying arbitrages opportunities (in the practical sense) if there is a discrepancy between the implied and the observed motion of the short rate.
- *Construction of market indicators*: The computation of the implied short-term rate process allows specifying the risk neutral density of probability for the short-term rate at some given future date. This density is deemed as a useful indicator by central banks, and is usually constructed from option prices. Using the process implied by the shape of the curve would be preferable, because all price or rates entering in a (liquid) curve are liquid, while options far from the money are not. In

depends on the parameters). We make use of this circumstance in the implementation of the calculus, as would be explained in Section 3.

addition, zero-coupon bonds need only a viable yield curve model to be priced, so the internal consistency of the method is not at stake.

- *Scientific interest*: It becomes worth investigating whether the implied and observed dynamics of the curve are consistent with each other. The coincidence between empirical yield curves and this specific functional form is by nature a cross-sectional or static phenomenon. It says nothing about the dynamic of the yield curves, and thus this finding is in itself not sufficient to validate the underlying arbitrage-pricing model as a description of the empirical reality.

Suppose now that the form is also complete. What does it bring?

- *Practical usefulness*: An option market maker may imply the values of the coefficients and then price – and hedge – options with it, exactly as a FX option market maker would imply the Garman-Kohlhagen volatility to do pricing and hedging. (The latter does so, even knowing that this volatility is not constant, hence the former can do so, even knowing that the implied coefficients are not constant)
- *Construction of market indicators*: If the form derives from a complete model, then the construction of a risk-neutral density function for the short-term rate can be refined, because some option prices can imply (or co-imply, together with the yield curve) the values of the parameter of the model. Concretely, a variety of options can be put to contribution: Near money options on the short-term future contracts, on the long-term future contracts, swaptions, caps and floors. The set of data supporting the method becomes extremely rich, information is presented in redundant ways – permitting a better control – and the internal consistency of the method is retained, owing to the completeness of the model. However, computations become heavy.

The particular functional form that we are now going to fit to the empirical euro and dollar yield curve is the DK1 form, given by formulae (13) or (14). It is altogether parsimonious, viable and complete

3 THE METHODOLOGY

In order to enable the reader to reproduce – or simply to judge – the results, a precise description of the methodology is useful. The present part has a technical scope, and it is subdivided in two headings: Data and algorithms. In both cases, one has to glance at specialised matters: Finance-related issues for what concerns the data, programming issues for what concerns the algorithms.

3.1 The data

The present section aims at specifying which financial data have been used, and how they have been organised. For what regards the three main currencies, the outstanding liquidity of unsecured money markets as well as of government bond markets qualifies those two market segments as appropriate data source, and we restrict ourselves to them. We begin with the description of the available data, specifying their number, the financial instrument they pertain to, and the range of dates they cover. Then we specify how those financial instruments have been attributed to various yield curves. Some financial instruments may belong to several yield curves.

3.1.1 The data set

In the case of the German mark / euro, the data set contains daily data covering over 200 instruments of all types. They include:

- money market rates (from the British Bankers Association (BBA), apart from the 1-day and 1-week rates),
- Euromark and Euribor future contract closing prices (the 20 available maturities),
- prices of German government bonds of all maturities (typically 80 prices at a given date)
- and swap rates (from 2-year up to 10-year).

In the case of the US dollar, the data set contains daily data covering over 400 US T-Bonds and T-Bills, and over 100 instruments of various other types, namely:

- money market rates (from the BBA, apart from the 1-day and 1-week rates),
- CME-Eurodollar future contract closing prices (the 40 available maturities),
- Federal Reserve Constant Maturities Treasuries yield of the 9 available maturities
- and swap rates (from 2-year up to 10-year).

In the case of the French franc / euro, the structure of the data set is similar to the one of the German case.

The Spanish data set contains only bonds and bills.

All data have been retrieved from Datastream, except bond prices that come from Bloomberg. They have been organised into 5 databases. The following table indicates the descriptive statistics of those 5 databases, namely the range of date they cover, the number of financial instruments they refer to, and the number of data. Note that there are two bases devoted to American instruments, named respectively BONDS and TBONDS. The former contains Datastream data on futures, swaps and CMT, while the latter contains Bloomberg data on US government dills and bonds.

Table 1

BASE	FROM	TO	INSTR.	DATA
BONDS	03-Jan-94	04-Jul-00	101	123967
BONOS	01-Mar-94	30-Jun-00	102	47254
BUNDS	01-Mar-94	30-Jun-00	263	239377
OATS	01-Jul-94	07-Jul-00	324	130700
TBONDS	01-Jul-97	04-Jul-00	514	214809

3.1.2 The yield curves

We intend to deal with the euro and the dollar, but have to use historical time series starting before the introduction of the euro. Owing to this, we will consider 4 currencies, the German mark, the French franc, the Spanish peseta, and the American dollar, and this will have a bearing on the curves that we can build. In the first two cases, we will build 3 yield curves, namely a government curve, a future curve, and a swap curve. Those are the core markets for German mark as well as for French franc. In the US case, we can add the Constant Maturity Treasury (CMT) curve, which has been built from the Fed's CMT indices, and contains therefore a relatively small number of data. In the Spanish case, we will only consider the government curve. This makes altogether 11 curves, 4 of them referring to the U.S., 3 of them referring to Germany, 3 to France, and one to Spain. (It is superfluous to say that the French and German swap curve become confounded after January 4th, 1999.)

The futures curves have been respectively made with CME Eurodollar future contracts and with LIFFE Euromark or MATIF Pibor and then Euribor future contracts. The swap curves are made with swaps against 6-month rates, even in the American case.

The American, French and Spanish government curves have been completed with short-term government notes (T-Bills, BTF and Letras del Tesoro.) In the French case, the data set encompasses only recent BTF prices. So the number of instruments included in the curve more than doubles in recent years, as can be observed on Figure 1b.

We will systematically complete the American futures curve, swap curve and CMT curve, as well as all the German curves, with overnight and Libor rates on their short end. This calls for a number of remarks. Such a completion can in principle create two types of problems, the first one being related to the loss of synchronicity of the data, and the second one being related (in the case of the German government and US CMT curves) to the loss of credit risk homogeneity. The first problem will be disregardedowing to the relatively low intraday variability of the deposit rates, as well as the second problem because of the relatively high bid-ask spreads on the short term deposit rates. This is certainly a lack of rigor. Nonetheless, it does not endanger the validity of the findings: While those imprecisions might have prevented us from observing some fine statistical regularity, they cannot fake a non-existing statistical regularity.

The following table summarises how the various data have been attributed to the 11 yield curves.

Table 2

BASE	CURVE	FROM	TO	INSTR.	DATA	DAYS
BONDS	US FUT	03-Jan-94	04-Jul-00	72	77757	1697
BONDS	US CMT	03-Jan-94	04-Jul-00	10	16970	1697
BONDS	US SWP	03-Jan-94	04-Jul-00	24	38373	1697
BONOS	ES GVT	01-Mar-94	30-Jun-00	102	44971	1654
BUNDS	DE FUT	01-Mar-94	30-Jun-00	52	33736	1650
BUNDS	DE GVT	01-Mar-94	30-Jun-00	181	169069	1649
BUNDS	DE SWP	01-Mar-94	30-Jun-00	24	39021	1649
OATS	FR FUT	01-Jul-94	07-Jul-00	48	27860	1567
OATS	FR GVT	01-Jul-94	07-Jul-00	243	64316	1567
OATS	FR SWP	01-Jul-94	07-Jul-00	26	37265	1567
TBONDS	US GVT	01-Jul-97	30-Jun-00	432	157278	779

For example, the last row of this table reads as follows: The data from the U.S government curve have been extracted from the database named "Tbonds", these data range from 1 July 1997 to 30 June 2000, they provide 779 observations of the curve, they include 157278 prices or yields of 432 different financial instruments.

One may incidentally observe, by summing over the column "Days", that the study has involved the computation and the fitting of no less than 17,173 yield curves. This stresses the key importance of algorithmics.

3.2 The algorithms

The present section takes a closer look at technical programming issues. The computation of a DK1-fitting curve requires a full range of operations, most of them deserving some explanations. In drafting this section much attention has been devoted to concrete and applied details, which are mainly of interest for a reader that would intend to perform similar analysis, (or to confirm or infirm the present analysis).

All the algorithms used can be implemented with an object-oriented compilable language, such as C++, Delphi or Visual Basic.

- The object-oriented feature is appropriate for the representation of a financial instrument (a list of cash flows, consisting of one numeric data and one date), a bootstrapped curve (a list of items consisting of one numeric data and one date, giving a computerised representation of a real-defined, real-valued function), or what could be called a "row" term-structure (by which a list of (priced) assets is meant, by opposition to the bootstrapped curve). A key point is that none of those lists can have a predefined size.

- In addition, the object-oriented feature is well appropriate allows for the representation of the mathematical relationships between financial instruments and yield curves. Those mathematical relationships as such can be explicitly implemented within the so-called "methods" of the objects. This way of doing alleviates the task of controlling and checking the source code for conceptual errors, as well as the task of maintaining that source code – at the cost, admittedly, of an increased computational burden.
- However, if one disregards the specific issue of the representation of financial instruments, then the necessity of an object-oriented language appears to be less compelling. In particular, the optimisation algorithms (by opposition to the bootstrapping and filtering algorithms) could clearly have been written in non-object-oriented languages.

On personal computers cadenced slightly below 1 GHz, using a Visual Basic code, the processing of a few years of daily data is a matter of hours.

3.2.1 Interpretation of the market data

The data can be clean prices or yields of bonds or bills, money market rates, and futures prices. They are available under a conventional form, which depends on the nature of the instrument and on the country. Those data contain information, but owing to the use of market conventions of quotations, this information is, so to speak, "encrypted". Thus interpreting the price or yield data implies to "decrypt" them, so as to exactly reproduce the cash flows involved in a trade executed at that given price or yield. A mistake in the "decryption" would result into an erroneous reproduction of those cash flows, which on turn would trigger an erroneous computation of the yield curve. One has to pay due attention to country specificities or to market segment specificities. In particular, it is worth mentioning that:

- The settlement day convention, which varies with countries, with market sub-segments and with the date (in the case of the 3 currencies that eventually became part of the euro) has been taken into account.
- Other national specificities, like the rounding of the accrual interest for the French OAT (but not BTAN), or the possibly missing first coupon of the Spanish Bonos, have also been taken into account.
- The impact of Saturdays and Sundays on actual cash flows has also been taken into account: Coupon bonds payments due on a weekend have been shifted to the following Monday. Repayments of money-market loans falling on a week end have been shifted to the following Monday, except when the following Monday fell in the following month, in which case the repayment dates were shifted to the previous Friday.

Only the impact of country-specific non-working days has been neglected.

3.2.2 Bootstraping

The method entails the recursive building of the so-called "bootstrapped curve" [9]. This is a zero-coupon rate curve constructed in such a way that

- it reprices all the instruments contained in the above described yield curves,
- and that it can be entirely described through a finite (though variable) number of figures (those figures being rates and dates).

Each iteration of the recursive algorithm builds the curve up to a further date. An iteration of the recursive algorithm consists mainly of a Newton method, perfectly similar to the familiar one that is used in the calculation of a bond yield. But the Newton algorithm is replaced by a closed-form expression whenever the instrument under consideration contains only one cash-flow whose pricing is not determined by the part of the curve that is already computed. This of course is dictated by computational efficiency: One can observe that this implementation has some bearing on it, insofar as it is used for large yield curves.

The following table illustrates the process and should allow one to "visualise" it. Let us consider as an example the bootstrapping of the German government curve as at the 15/04/1998. (The resulting curve is the one shown in Figure 3 in 5th position.) The table displays 18 financial instruments, and it displays them in the order in which they are actually encountered during the calculation process. (This does not mean that the instruments shown in the table are the first 18 instruments encountered during this calculation: In order to make that table fit one page, it is necessary to skip some of them.) The headings of the *columns* refer to those 18 instruments. They consist of five money market rates, the overnight ("ON"), the tom-next ("TN"), the 1-week ("SW"), the 1-month ("1M"), the 2-month ("2M") and 13 bullet bonds (referred to as "B")[10]. The dates of the financial flows involved by those instruments are the headings of the *rows*.

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9 There is no relation with the homonymic statistical technique.

10 By now, the use of OIS rates instead of deposit rates would be more natural. But this would have restricted the euro sample to the 18 months covering 1999 and the first half of 2000.

Table 3

Date	ON	TN	SW	1M	2M	B	B	B	B	B	B	B	B	B	B	B	B	B
16/04/1998	3.523																	
17/04/1998		3.504																
20/04/1998										3.459								
23/04/1998			3.447															
29/04/1998											3.526							
18/05/1998				3.585														
20/05/1998												3.589						
28/05/1998													3.601					
17/06/1998					3.617													
18/06/1998														3.618				
22/06/1998															3.623			
01/07/1998																3.631		
15/09/1998																	3.663	
17/09/1998																		3.663
20/11/1998						3.672												
02/12/1998							3.673											
17/12/1998								3.684										
18/12/1998									3.679									

The computation of the rates shown in the 9 first columns of the table is straightforward. The 9 last bonds have a cash-flow (a coupon) falling into the range of the already computed curve. It is necessary to compute the zero-coupon rates for those intermediate dates before computing the zero-coupon rates for those maturities, since the later depends on the former. The zero-coupon rates for those intermediate dates are the ones that are displayed in those 9 last columns of the table.

One will observe that the computation of the zero-coupon rates for those intermediate dates involves a convention of interpolation. For the sake of simplicity, it has been assumed that the instantaneous forward rate was constant between two dates where the zero-coupon was explicitly calculated. Other choices would have been possible. It has been checked – though not in a systematic way - that a change in this convention did not affect the result. To be specific, two German government curves computed from the same data but with two distinct interpolation conventions cannot be visually distinguished.

3.2.3 Filtering and completion.

In all cases but the US swap curve, CMT curve and eurodollar curve, a *filtering* against possible erroneous data has been applied. The instruments responsible for an excessive irregularity of the bootstrapped yield curve are cancelled out and the curve is recomputed without considering them. The filtering process consists in successive executions of those removals of financial instruments and curve buildings. It ceases when the obtained yield curve is smooth enough. At this point, the repricing of any remaining financial instrument with this curve gives back the original price of this asset. Typically, this filtering procedure eliminates only a few instruments. In most cases, they correspond to obvious mispricings. Those mispricings can be due to plain data errors or to the incorrect specification of the instrument's quoting convention or cash flow. For the purpose of illustration, let us mention the case of the Spanish government bonds. Those treasuries usually do not pay a coupon the first year: Ignoring this particularity would result into an erroneous representation of their payment schedule and would produce a "kick", that actually does not exist, in the bootstrapped zero-coupon curve. Other misspecifications of financial instruments repayment schedules, due to the wrong understanding of market conventions, might produce similar deformations of the curve. Such deformation, though possibly small in absolute terms, should be avoided in a study whose purpose is to recognise whether the curve does follow some functional form.

On the top of that, a *completion* procedure has systematically been used. Curves are computed for successive working days. If the curve from day D reached maturity T_1 and the curve of day D+1 reached maturity $T_2 < T_1$, then the second one was completed between T_2 and T_1, keeping constant the forward rates between T_2 and T_1. The completion procedure aims at avoiding outliers due to missing data.

3.2.4 Estimation of the parameters

The parameters of the model are estimated via an elementary method based on grid searches.

The basic idea is as follows: One considers regularly spaced values for each parameter a, b, μ, ν, and for the state variable r. For every possible combination, the theoretical yield curve is computed, and then compared to the bootstrapped empirical yield curve. The values of the parameters and state variable producing the closest theoretical curve are kept. They are used to start a new grid search with smaller discretisation step.

The method involves a small knack, though. The metric defining the proximity has been made insensitive to parallel shifts. This means that each grid search produces a theoretical yield curve that is the closest to the actual one, but only up to a (possibly important) parallel shift. A closed-form transformation of a, b, μ, ν and r finally allows to correct for it [11]. The aim of this refinement is to better exploit the information provided by the numerous computations that have to be done anyway. In essence it replaces implicitly redundant calculations with similar, but non-redundant ones: This does not diminish the computational burden of the grid search, but increases its payoff and therefore is instrumental in shrinking the overall ratio between accuracy and speed.

Apart from this small refinement, the method is quite simple. More sophisticated ways like gradient methods have been tested, but without success. This could be due to the strong non-linearity of the mapping between the 5-uplet (a, b, μ, ν, r) and the corresponding theoretical curve.

[11] This trick ultimately relies on that mathematical property, to which was referred in Note 7, that DK1 curves are CIR curves plus constant.

4 THE RESULTS

The main finding of the paper is the extreme goodness-of-fit of theoretical curves based on the particular functional form that the DK1 model prescribes. In short, the data abundantly provide DK1-shaped yield curves. Clearly the precision - or lack thereof - of the fit is essential to assess whether this "DK1 shaping" has a meaning at all, or whether it is the mere result of chance. We will therefore first examine how good was this fit, and how it evolved, during the last six years. As those observations would be of limited interest if they were too dependent of the goodness-of-fit measure, we will also examine in more detail the notion of goodness-of-fit in term of topological distance. The results won't be affected by reasonable changes in the definition of the goodness-of-fit measure. We will examine the evolution of the parameters of the fitting curves.

4.1 Evolution of the goodness-of-fit over the six last years

We measure the goodness-of-fit of a theoretical DK1-curve to an empirical bootstrapped curve via a specific measure that can be called the L^2-measure. It is defined as the square root of the average of the square of the difference of the two curves, for maturity ranging between 0 and 10 years [12]. It is therefore expressed in basis points, as are the yields themselves. (As was previously said, alternative measures of the goodness-of-fit will be considered below: Those alternative measures too will be expressed in basis points.)

The behaviour of the goodness-of-fit over the six last years exhibits different features in the American case and in the European one.

This behaviour is displayed in Figure 1. The four charts on the left column of Figure 1 refer to Europe, while those on the right side refer to the U.S. In each of those eight charts, the black line shows the goodness-of-fit measure expressed in basis points (left scale). The grey line indicates the number of financial instruments contained in the curve (right scale), as this information is of key importance to assess the meaning of a good fit.

4.1.1 The American curves

Four American curves have been constructed: The government curve (circa 200 T-Bills and T-Bonds), the CMT curve (10 CMT indexes), the future curve (a few short-term deposit rates and about 40 eurodollar future contracts), and the swap curve (deposit rates and swap rates, about 20 rates).

[12] In mathematical terms, it is therefore the L^2-norm of the difference of the two curves, defined as functions of the uniformly measured 0-10-year interval

As one can observe on the American charts of Figure 1, the L^2-measure stays for most of the time below five basis points. Logically, better fits can be obtained when the curve contains only few financial instruments: Consequently the US CMT curve (10 instruments) consistently exhibits a fit of a mere 2 or 3 basis points. But richer curves like the US futures curve (circa 50 instruments) or government curve (circa 150 instruments) also exhibit rather accurate fits.

Two observations are noteworthy.

- The two monetary curves, (the futures curves and the swap curve,) display three periods of (relatively) bad fitting. Those periods last between one month and one quarter. They coincide for the two curves. The first one is around July 1995, the second one is around end 1995, the third one is around end 1999. By contrast, the two riskless curves, (the government curve and the CMT curve,) have a stationary behaviour.
- The US government curve's goodness-of-fit exhibits a slight, but clear trend. This trend goes towards a worst fit and is accompanied by a decline in the number of instruments (here, T-Bills and T-Bonds) present in the curve. In 1997, the typical fit is around 5 basis points, for roughly 175 instruments, in 2000, the typical fit is around 7 basis points, for roughly 125 instruments. (Notice however that a parsimonious model giving the price of 125 instruments up to 7 basis points can still be considered outstanding.) By contrast, the fit of the three other curves does not show any trend.

4.1.2 The European curves

Seven European curves have been constructed. Two of them, however, are not likely to bring useful information. They are the ones constructed with futures, for France and Germany. Unlike in the U.S. case, the future curves cover only the five first years of the maturity segment, moreover it should be added that this is a recent feature and that the fourth and 5th years can still be seen as illiquid. We will therefore focus on the three government curves (France, Germany, Spain) and on the German swap curve.

The German swap curve is fully comparable to its American counterpart, being made with more or less the same number of instruments. One can observe that it reaches comparably good fits only during the three last years, with the exception of a period of several months encompassing the launch of the euro.

For what regards the government curve, it is striking that a fit closer than five basis points has been reached much sooner in France than in Germany. As a foretaste of the interpretation of the broad phenomenon, we can notice that more technical, or scientific, trading strategies, already prevailed in France in the early nineties, but spreaded only recently to the rest of Europe. The results of the French government curves are comparable to those of the U.S. futures curves: Same level of fit (around or below five basis points), same number of instruments (circa fifty), and even same dynamics if one neglects the U.S spike of January 1999. The dynamics of the goodness-of-fit for the German government curve are similar to those of the German swap curve and display decent results only from 1998 onwards. It is consistently excellent in 2000, with fit ranging between 3 and 5 basis points (for a curve containing circa 80 instruments!) which is much better than its French counterpart.

As regards the Spanish government curve, one can observe that the fit is not so good, even in the last years, even if it remains somewhat reasonable from 1998 onwards (circa 10 basis points). The increase in the number of instruments during the two last years reflects only the availability of Spanish T-Bill data.

The general impression is firstly that the goodness-of-fit converges towards its American level, secondly that this convergence started first in France, and that the German curves took more recently the lead.

4.2 About good fits

In itself, a given goodness-of-fit criterion does not need to encompass fully the DK1-shape stylised fact that has been evidenced. For if that fact holds true, then it must be robust: This means that the particular choice of a given goodness-of-fit criterion should not matter. Thus, we replace the L^2- with two other ones, so as to see whether it changed the picture: The picture does not change. Even so, though, the goodness-of-fit is just a figure and numerical results do not speak to intuition. This is why we should devote some attention to a few number of curves where the quality of the fit is extraordinary. A graphical examination of those "hits of the fits" tells more than any sample of figures. In addition, this naturally gives a first hint about what could be the interpretation of the phenomenon.

4.2.1 A good fit does not depend on the choice of the criterion

We have chosen the L^2-measure, but we could also have chosen the L^1-measure or the L^∞-measure. The L^1-measure means the average of the absolute value of the spread between the theoretical and the empirical curve. The L^∞-measure means the maximum value of this spread. It is natural to start with the L^2-measure because it is, in some sense, the centre of a class of metrics ranging from the L^1-measure to the L^∞-measure. However, it is tempting to see what happens when replacing the L^2-measure by the L^1 or the L^∞ one. Figure 2 displays the results. (Note that the graphs of that Figure 2 have logarithmic scales.) Let us summarise this answer by noting that:

- The dynamics are the same,
- the numerical estimation of the goodness-of-fit remains practically unchanged when one replaces the L^2-measure by the L^1-measure.

That the L^∞-measure takes higher values is not very surprising, since this measure can be contaminated by a high difference due to one financial instrument only. This makes also the L^∞-measure less appropriate than the two other ones.

4.2.2 Some quasi-perfect fits

We selected, among the 17,173 yield curve estimations, 9 for which the fits are nearly perfect. The graphs showing the empirical bootstrapped curves and the fitting DK1 curve can be found in Figure 3. To fully appreciate those results, one should always remember how many instruments prices are explained by just five numbers: This is the reason why this number of instruments appears on the charts contained in Figure 3.

4.2.3 A first glance at the interpretation of the goodness-of-fit phenomenon

Having seen Figure 3, one is naturally led to suspect that the origin of the phenomenon might well involve the use of computers: Whatever might have led the asset prices to reproduce the mathematical form of a DK1 curve, it has implicated intricate computations.

Of course, this is not proved yet. In principle, it might also be that the complicated interaction between the market participants tends, for some hidden mathematical reason, to produce this particular form of curves. This seems a little bit difficult to accept. Except, however, in one case: One could imagine that the complicated interaction between the market participants tends to produce a certain term structure motion, and that this term structure motion is precisely the one that the DK1 model postulates.

This hypothesis is not unreasonable. Nevertheless, the results that we will get on the DK1 model parameters do not support it. The DK1 model potentially provides a consistent explanation of the DK1 shape of the curves, but this explanation relies, among other things, on the fact that the parameters stay constant. By its very logical structure, the explanation actually uses this hypothesis that a, b, μ, ν are constant (and hence deterministic), and consequently explanation does not work anymore when this hypothesis does not hold true. Now, the fact that a, b, μ, ν stay constant or do not can be readily obtained from the data. The answer is univocally negative, as shows Fig. 5.

4.3 Implied parameters of the DK1 model

Having observed that a model fits the data, one is naturally led to examine the values of the model's parameters for which the data are matched. For doing so, we need to introduce a subtle distinction. The model of the empirical curve is the particular DK1-curve that happens to fit with it. This particular DK1 curve is given by five numbers. Among those five, four (a, b, μ, ν) are the parameters of the DK1 model, the 5^{th} (r) being the state variable or factor. we will refer to the four ones as the DK1 implied parameters.

4.3.1 The implied parameters on the long run

Over long periods, the implied values of parameters display an utterly erratic behaviour. A simple glance at Figure 5 is enough to be convinced that none of those four parameters is stable over time. This holds true also for the curves that have achieved a good fit. Consequently, a yield curve that was accurately approximated by a DK1 curve at a certain date can be also accurately approximated at another date by another DK1 curve. But the two DK1 curves won't have anything in common – except the naked fact that they are DK1. Therefore, even when the DK1 curves fit the empirical ones, the DK1 model is not an accurate description of the reality – since it is consistent only with constant values of its parameters.

4.3.2 The implied parameters on the short run

By contrast, over shorter periods, some curves produce a relatively more stable set of parameters. This observation suggests that it is feasible to use the DK1 model, once calibrated on the empirical curve, to compute the value and hedge ratios of complex market positions, including options and other derivatives besides simple bonds and loans. Thus, in some cases one can use the DK1 model as an arbitrage pricing model. Essentially, what can be done in those cases is either to price complicated claims in a way which is consistent with the observed curve, and/or to compute hedge ratios (deltas) both for those complicated claims and for the more simple instruments that are included in the curve.

4.4 Implied value of the short-term interest rate

The DK1 curves are specified by five numbers. Four of them are the parameters of the model itself. The 5th is the factor itself, which in the particular case of the DK1 model is simply the overnight interest rate. This 5th number is special: Not only does it have an implied value, as the four model parameters a, b, μ, ν, and as would have the volatility in the case of a Black and Scholes, or Garman and Kohlhagen, model. But it also corresponds to a readily observable value, which is the money market overnight rate.

The overnight rate exhibits a strong day-to-day volatility. By contrast, the implied value for r is much less volatile. Therefore, it might make sense to compare the implied r to the policy rate rather than to the overnight market rate.

Figure 6 displays the result in the case of the US and of Germany. The policy rate is the Fed target rate in the case of the US and the Bundesbank repo rate (before 1999) and ECB refi rate (from 1999) in the case of Germany. As one can readily check, the implied r sticks to the policy rate whenever the empirical curve is well fit by the DK1 curve, which means all the time in the US case and from 1998 onwards in the German case.

4.5 Interpretation of the goodness-of-fit phenomenon

After summarising the findings, it remains to try to explain them. This will drive us to interpret the goodness-of-fit as an indicator of the liquidity and – maybe more importantly - of the possible benchmark role of a yield curve.

4.5.1 Summary of the findings

We now gather the observations and attempt to summarise them into a consistent story.

- The first element is that the coincidence between the DK1-curves and the actual one is too accurate as well as too frequent to be the mere happenstance. In other words, the DK1 model exhibits some sort of a limited predictive power, and namely: In most cases, it correctly predicts the functional form of the curves.
- The second element is that the DK1 model does not describe the reality as would a physical model do. This follows clearly from the variability of its parameters, that this model assumes as being constant. (This may sound familiar. The DK1 model is therefore placed in the same situation than the Black and Scholes model or the Garman and Kohlhagen model when they are compared with empirical data: Those models imply varying values for their parameters, while - according to themselves - the parameters should be constant.) In other words, the DK1 model exhibit no complete predictive power: It falsely predicts that a, b, μ, ν remain constant.

4.5.2 Interpretation of the findings

The realisation of yield curves having DK1 form cannot be attributed to the correctness of the DK1 model. So the question is: What could it be actually attributed to?

We may describe that situation by saying that we observe a mathematical regularity that we cannot attribute to a cause. Such a situation, for sure uncomfortable, is not entirely new. There is another case of such an "over-mathematisation", which is well known if not well explained: The case of the chartist techniques. One faces there a similar puzzle. The mathematical regularities are observed, because chartists techniques work, but those regularities cannot be derived from a theory. In this case, the explanation looking the more convincing is that chartist beliefs are self-fulfilling. We refer the interested reader to Godechot (2001) pp. 218-230. If one accepts the parallel between the chartist puzzle and our DK1 functional form puzzle, then a possible explanation might be that the DK1 shaping is (results from) a self-fulfilling belief.

A possibility would be that functional form is already present in the pricing algorithms used by the market participants. Some of them can use the CIR model, and allow to modify it with a parallel shift – so to produce a DK1 model. It can also be reproduced by a discrete construction à la Ho and Lee. Or the model could be explicitly programmed in the computers of option traders.

This is only a possibility. Clearly one could not prove it without having access to the spreadsheets, macros and programs that market makers and traders actually use. The action of self-fulfilling beliefs, whereby if traders believe that the curve is DK1, then their own quoting and trading is bound to reinforce the DK1-shaping of the curve, could naturally be proposed as an alternative explanation. The beauty of this alternative explanation is that the DK1 shaping need not be explained by objective reasons (reading for instance "the DK1 model correctly describes reality"). The weakness of this alternative explanation is that it says nothing about the origin of the DK1 shaping. Thus, albeit one could believe in that power of self-fulfilling beliefs, we will tentatively adopt the first possible explanation, that prosaically claims that the DK1 form is somehow present in the pricing algorithms implemented on the traders computers.

What we can do is to read again, keeping this possible explanation in mind, the history of the six last years, so as to assess whether, and to which extent, it might make sense.

During the six last years, one observes that:

- the US curves are consistently well-fitted over the whole period,
- the French curves are correctly fitted over the whole period,
- the German curves are poorly fitted at the beginning of the period, but the DK1 shaping gains accuracy over the period, at the end their goodness-of-fit overcomes the French one and reaches the US one,
- the Spanish curve never has an excellent fit.

According to the author's personal opinion, the correct reading of the story should be as follows:

- At the beginning of the nineties, computerised and mathematised trading is already the rule on the US liquid curves, the government curve and the money-market curves. All the prices of the instruments that belong to those curves are derived from a small number of main prices –say, the 10-year, the 5-year and other important maturities. From this small number of main prices – likely 4 or 5 – the market makers obtain the other ones by applying mathematical rules. Those rules do not need to be derived from a model representing accurately the reality. (Maybe should we underline that this style of pricing makes sense only in the case of very liquid financial instruments. Indeed, without narrow bid-ask spreads, the precise computation, by sophisticated methods, of prices does not make a lot of sense.
- At this point of time, this style of pricing gains importance in France. It is favoured by the typical scientific background of the traders of the Paris market place. This style of pricing generates quantitative trading, which is in itself a source of liquidity. Not surprisingly, it corresponds to the success period of the Notionnel future contract. The trend toward smoothness of yield curves, which seems to have originated in the American market, starts to affect the French market.
- Around the middle of the nineties, the quantitative-type of pricing reaches the German market. This might have been triggered by several factors. One of them is the predominant role of the

London financial market, which rules the European interest rates. It becomes convenient to price all curves in spread to a particular one. The German curve takes naturally this role of reference curve because of the size of the German economy. Also, the predominance of the London place favours an unification of the methods in use in the financial industry, and gives an incentive to Deutsche Mark traders to adopt the quantitative style of pricing. Another thing is that the relative success of the 5-year Bobl future contract makes the German curve even more adapted than the French curve to this style of pricing. This period sees an increasing success of the German 10-year Bund future contract, which challenges the French Notionnel.

- Starting from 2000, the German curve has become the European reference curve, as indicated by the volumes and open interests of the various German future contracts, or the decrease in the typical bid-ask spreads of German –but not French – government bonds. Quantitative pricing makes only sense for liquid assets, and makes more sense for more liquid assets. Consequently, it starts to be primarily utilised for the German curve.

In my view, the contagion of the DK1 form across the Atlantic cannot have originated from the mere linkage between markets: one could hardly conceive, through which concrete mechanism this somewhat subtle mathematical pattern could have been transmitted. However, one could more easily conceive that it results from some sort of self-fulfilling belief, or opinion, about – to state it vaguely "how the curve should be". Therefore, the emergence of this recognisable functional form for liquid yield curves should indeed reflect the way and manner prices are generated in modern trading. This way and manner hinges on the use – and abuse, could one suspect, - of mathematical tools

Fairly obviously, it can not be taken for granted that the use of the DK1 model or shape by the pricers of markets participants is deemed to result in the actual production of a curve having the DK1 shape. The mere possibility – to say nothing about the actual existence – of such a propagation mechanism would deserve to be substantiated by a heavy work of computer simulations. Despite the lack of it and in regards of the results, one is bound to think that the DK1 shape must arise from somewhere. This explanation thus appears as being the most likely, or the less unlikely.

This mathematical way of pricing exists mainly for liquid and benchmark curves. Consequently, the good fit to the DK1 functional form should indicate both liquidity and benchmark status of the curve (as a whole).

This explanation makes understandable why mathematical forms could be displayed by the yield curves. However, it does not explain why this particular mathematical form happens to emerge and to predominate. One could conjecture that this is due to some properties of that DK1-shape, so for example its smoothness, its concavity, the existence of a horizontal asymptote. But this is vague, and it is probably more honest to say that the "DK1-shaping" remains a puzzle.

5 CONCLUSION

Understanding how the term structure of interest rates is formed and how it behaves has always been regarded as a stimulating enterprise. This study investigates whether yield curves built from liquid instruments match the *pattern*, or the *form*, predicted from the Duffie and Kan 1-factor model (DK1 model). The DK1 model is derived on the basis of the no-arbitrage hypothesis and of the state variable (or factor) hypothesis. It encompasses the celebrated Cox-Ingersoll-Ross (1985) model (CIR model) and is encompassed in the not less celebrated Heath-Jarrow-Morton (1992) model (HJM model).

In carrying out this analysis, we have separated the static predictions of the DK1 model from its dynamics prediction. Predicting that the yield curve matches a given form is a static prediction, while predicting something about the motion of the yield curve is a dynamic prediction. The study establishes that the DK1 model performs well for the static part, and bad for the dynamic part.

This is somehow amazing. The DK1 model builds a consistent paradigm, which cannot be separated into a dynamic and a static part. One could have expected that it performs well for none the two things, or that it performs well for each of the two things. This observation suggests a rough explanation. In essence, the pricing methods used by market participants could explain the phenomenon.

So the quality of the fit of empirical curves by the DK1 curves should be related to its benchmark status. Consequently, one can estimate in a quantitative way which curve is used as a reference for the pricing of less liquid assets, and one can rebuild an history of the benchmark status. This is of particular importance in the case of the euro-area interest rates markets, where the old rules, references and traditions have become obsolete with the monetary union, and where the formation of the benchmark status is a still ongoing process.

The finding raises a number of starting points for further inquiry and investigation. In particular, it would be worth to reproduce the experiment on tick-data, because they are free from the observational noise, present even in the better daily data series, which arises from the non-contemporaneity of the data. Another appealing extension of the study, technically more difficult, would be the derivation of an implied risk-neutral distribution function of the short-term rate at future dates from the process that is implied by (DK1-shaped) yield curves. This would make sense, as the DK1 model is complete, and would rely on numerous and liquid data because the government money yield curves of the major currencies offer plenty of liquidity points. Finally, other examples of “over-mathematisation” of the structure of the yield curve could be tracked.

Annex

Proving that the DK1 model is actually a particular case of the HJM model amounts to checking that the instantaneous forward rate of the DK1 model is indeed solution of an integro-differential equation. Such a computation is not difficult from a conceptual point of view, but it is tedious. As the role of this computation is ultimately to bring a proof, it is important to ensure that, however tedious, it has been correctly performed. This makes a case for using a formal calculus software.

This Annex provides an example of an automated verification that the forward rate does solve the integro-differential equation. The verification is performed on a software enabling formal calculus (in this case, Mathematica).

In[4]:= `ClearAll["Global`*"]`

In[5]:= `(* I  The DK1 Model *)`

`(* I .1 the forward rate in in the DK1 Model *)`

`(* I .1.a  Definition of f in the DK1 Model *)`

$$f_{DK1}[r\_, t\_] = \left(\frac{2\gamma}{(\gamma+b)\,(E^{\gamma t}-1)+2\gamma}\right)^2 E^{\gamma t}\, r + 2\,E^{\gamma t}\gamma\,\frac{(E^{\gamma t}(\gamma+b)^2(b\mu^2+a\nu^2)+2(\gamma+b)\mu^2\nu^2+2a\nu^4)}{((\gamma+b)\,(E^{\gamma t}-1)+2\gamma)^2\,\nu^4} - 2\left(\frac{a}{\gamma-b}+\frac{\mu^2}{(\gamma-b)^2}\right);$$

In[6]:= `(* I .1.b  f in the DK1 Model solution of the HJM integro-differential condition  *)`

$$\left(\frac{\mu^2+r\nu^2}{2}\left(\int_0^t(\partial_r f_{DK1}[r, s])\,ds\right)^2 - (a-b\,r)\left(\int_0^t(\partial_r f_{DK1}[r, s])\,ds\right) + \int_0^t(\partial_s f_{DK1}[r, s])\,ds\right) \;/.\; \gamma \to \sqrt{b^2+2\nu^2} \;//\; \text{PowerExpand} \;//\; \text{FullSimplify}$$

Out[6]= 0

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Figure 1 a - Spread between empirical curves and their DK1 theoretical fitting curves

(left axis: spread in percentage points, right axis: number of instruments in the curve)

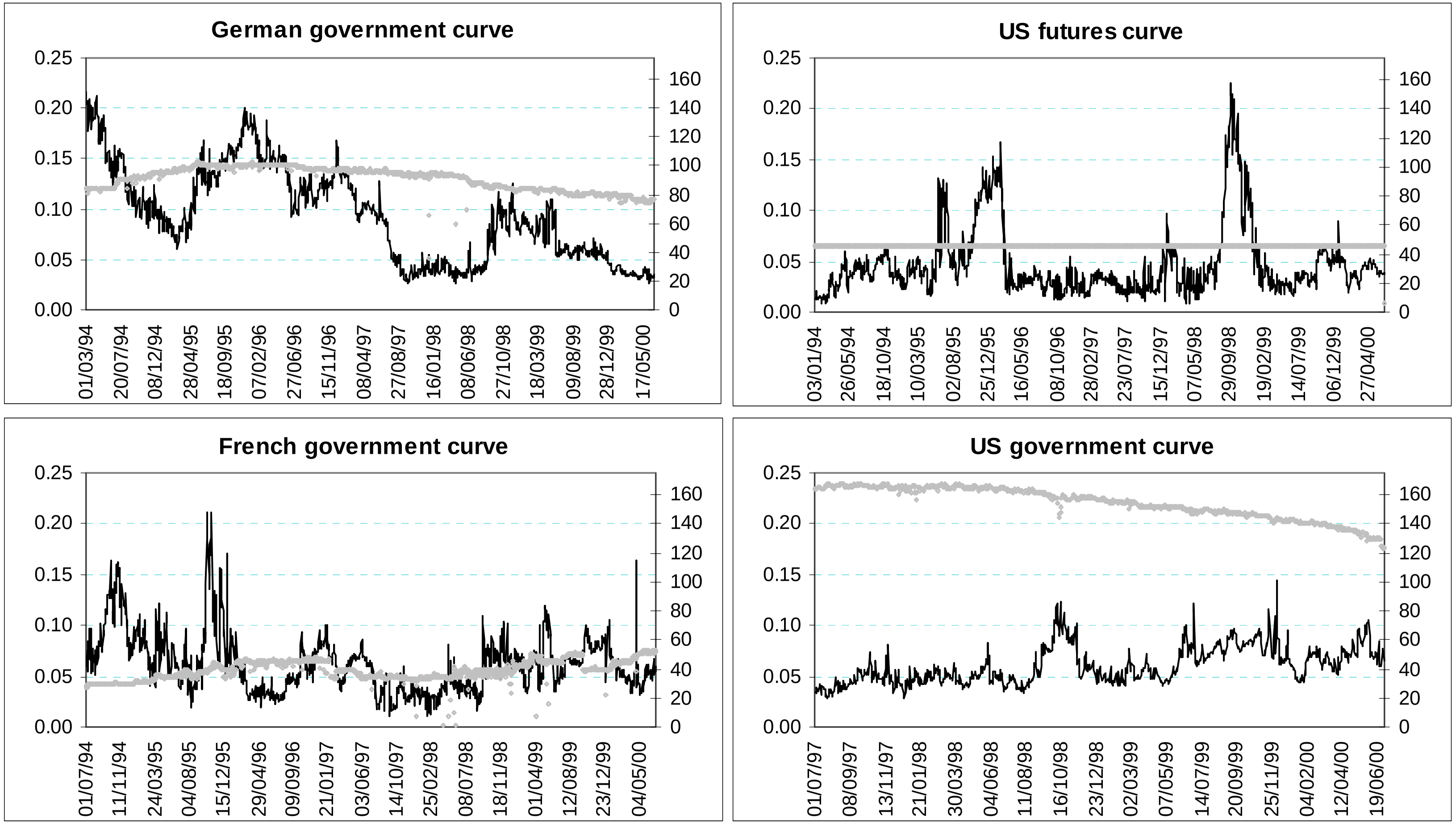


Figure 1 b - Spread between empirical curves and their DK1 theoretical fitting curves

(left axis: spread in percentage points, right axis: number of instruments in the curve)

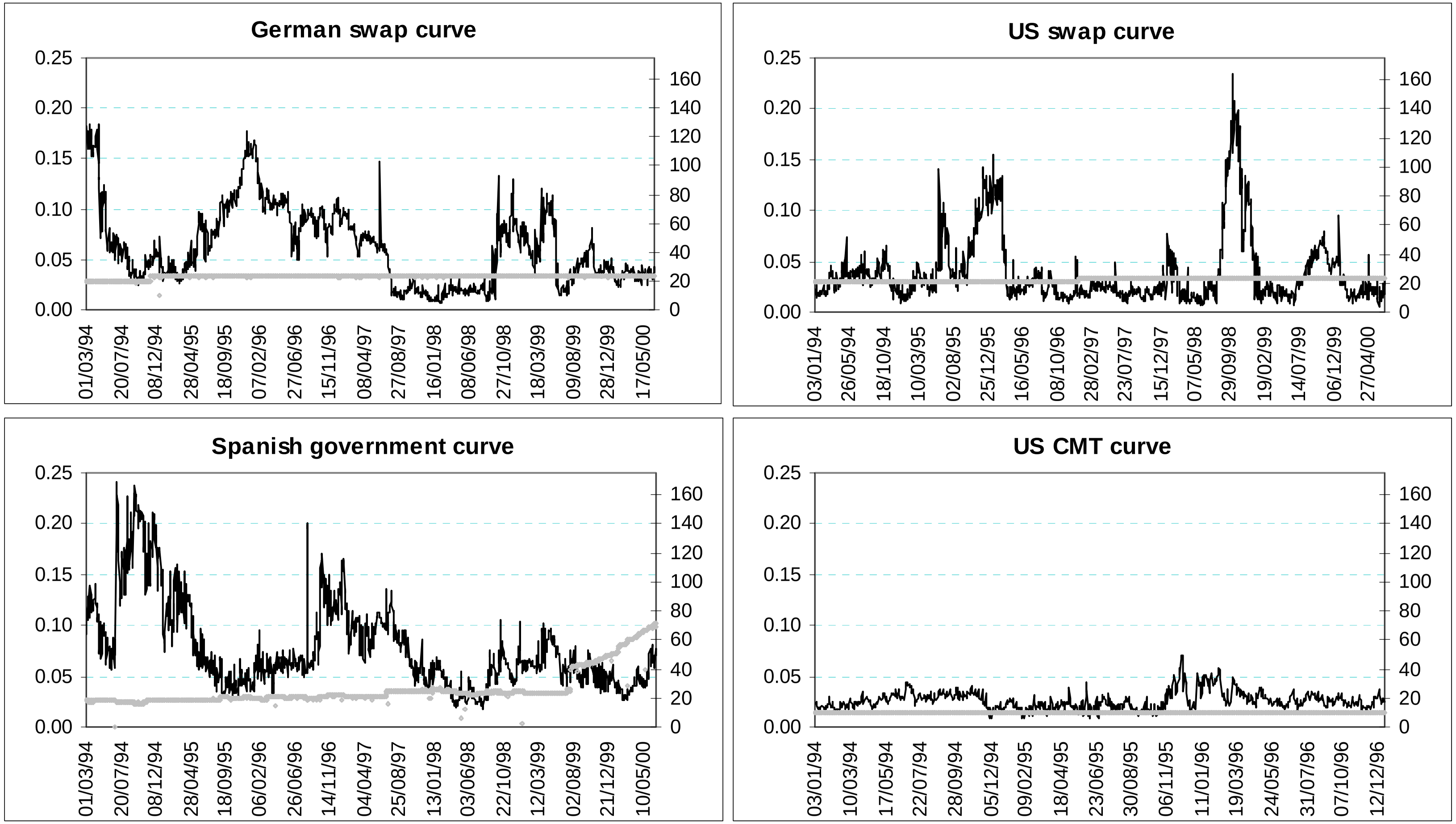


Figure 2 - Comparison between three measures of goodness-of-fit

(all spreads in percentage points, lower spread: L^1 , middle spread: L^2 , higher spread : L^∞)

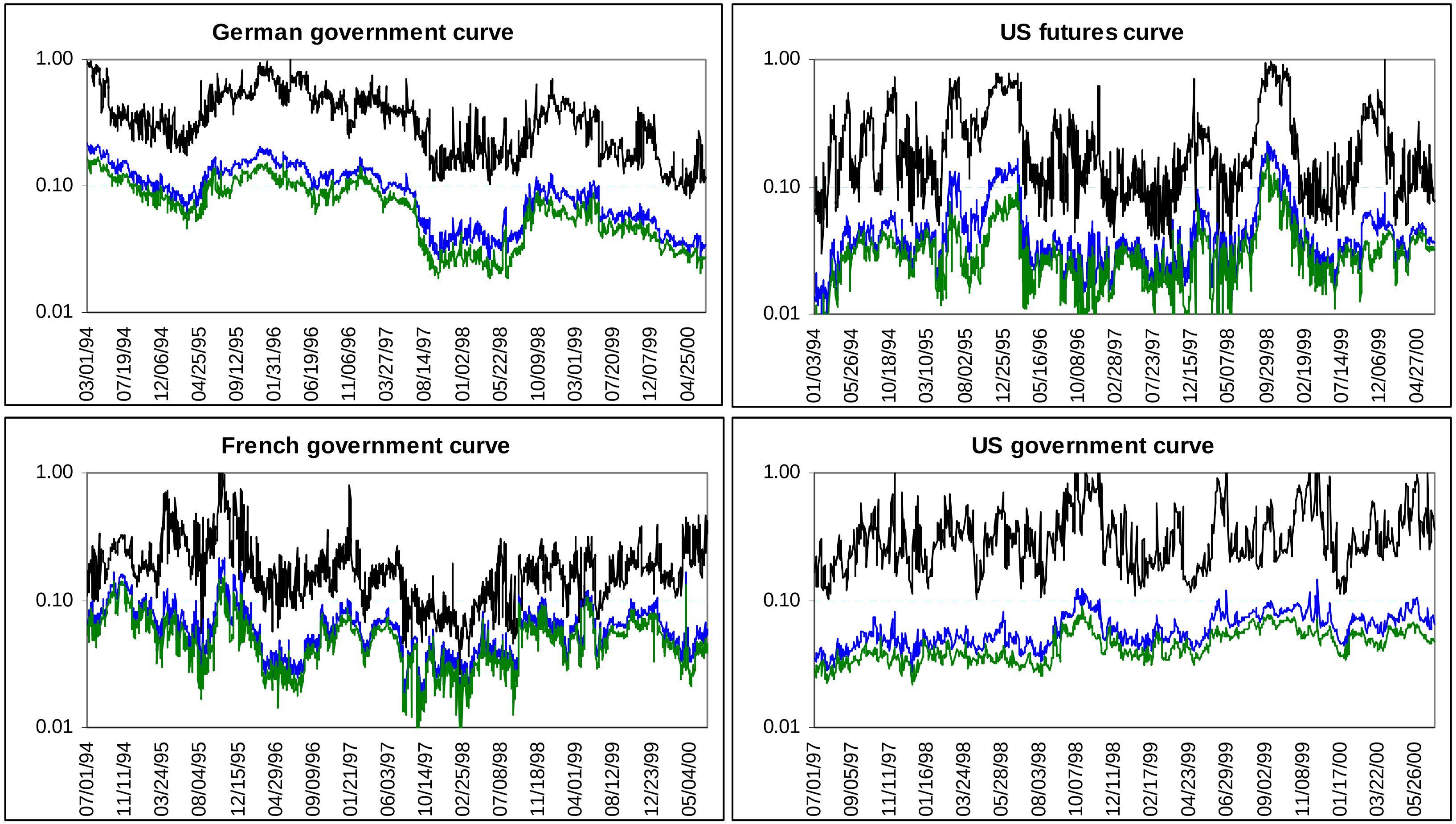


Figure 3 - Some examples of very good fit

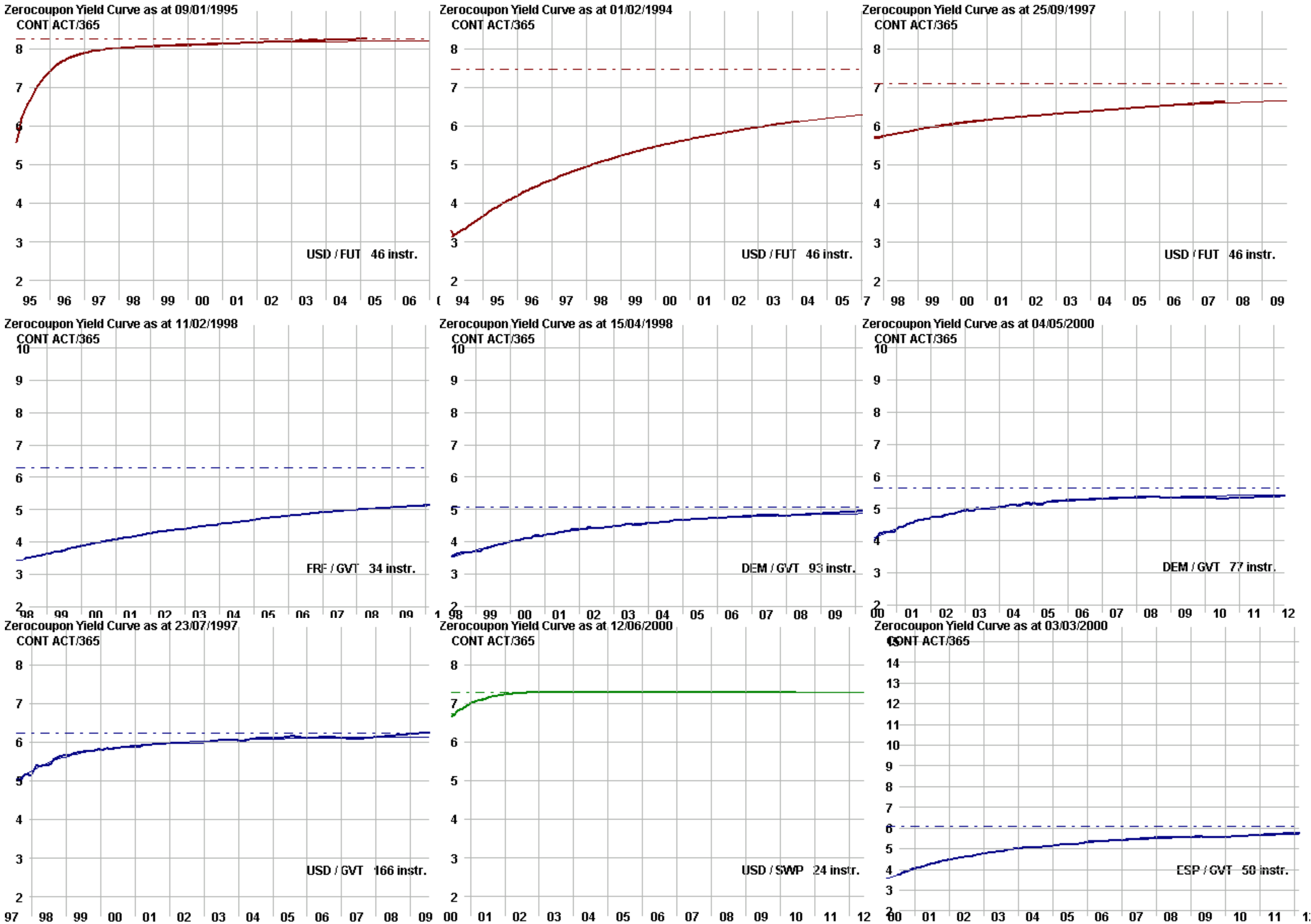


Figure 4 - Number of curves where the spread L^2 is below a given threshold

(white area: no data, light grey: less than 20 curves, possibly 0 curve,
darker levels of grey: between 21and 40, between 41 and 60, over 60 curves)

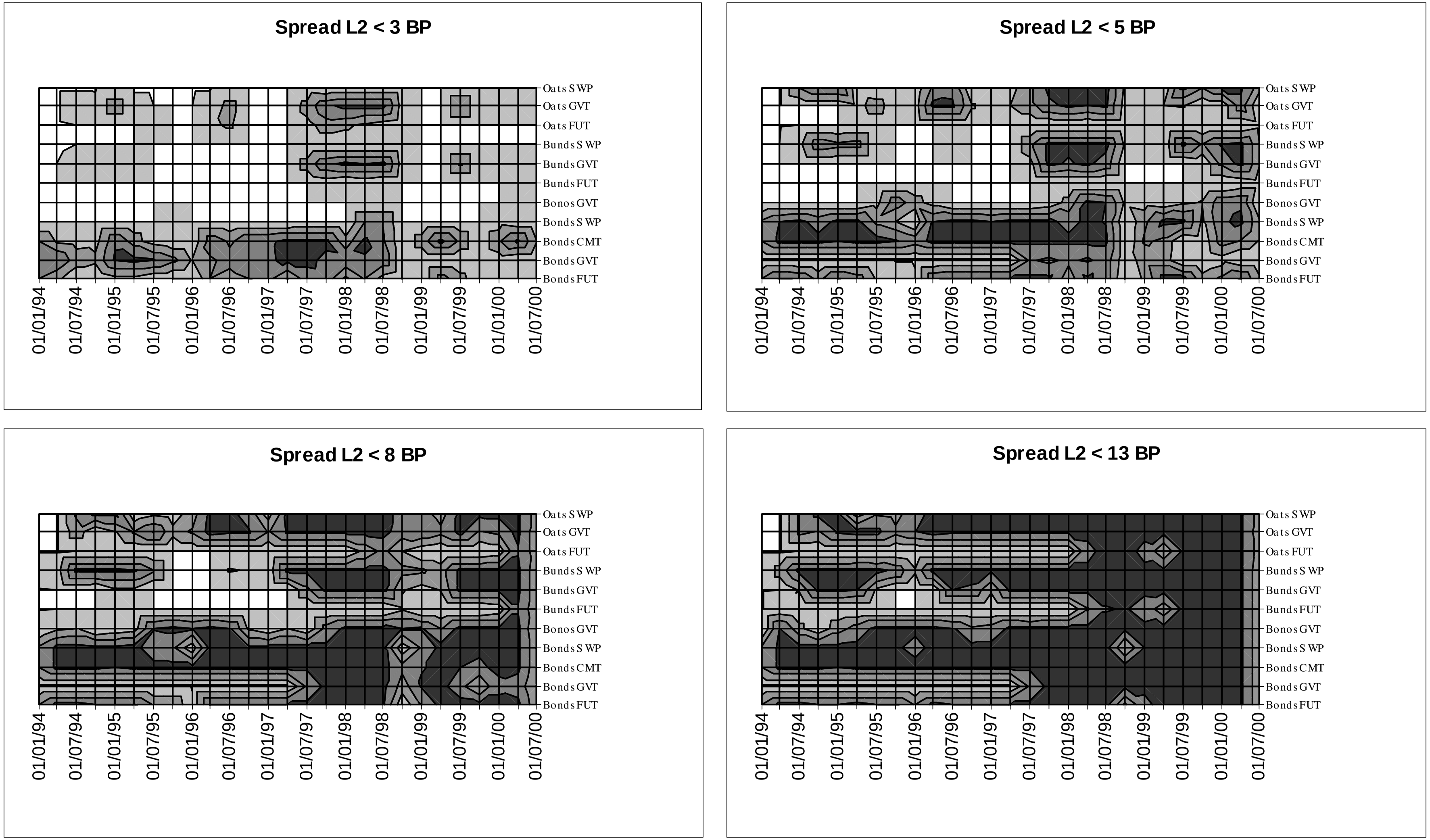


Figure 5 a - Estimates of the parameters of the DK1 model

(thin lines, left axis: b and ν, bold lines, right axis: a and μ)

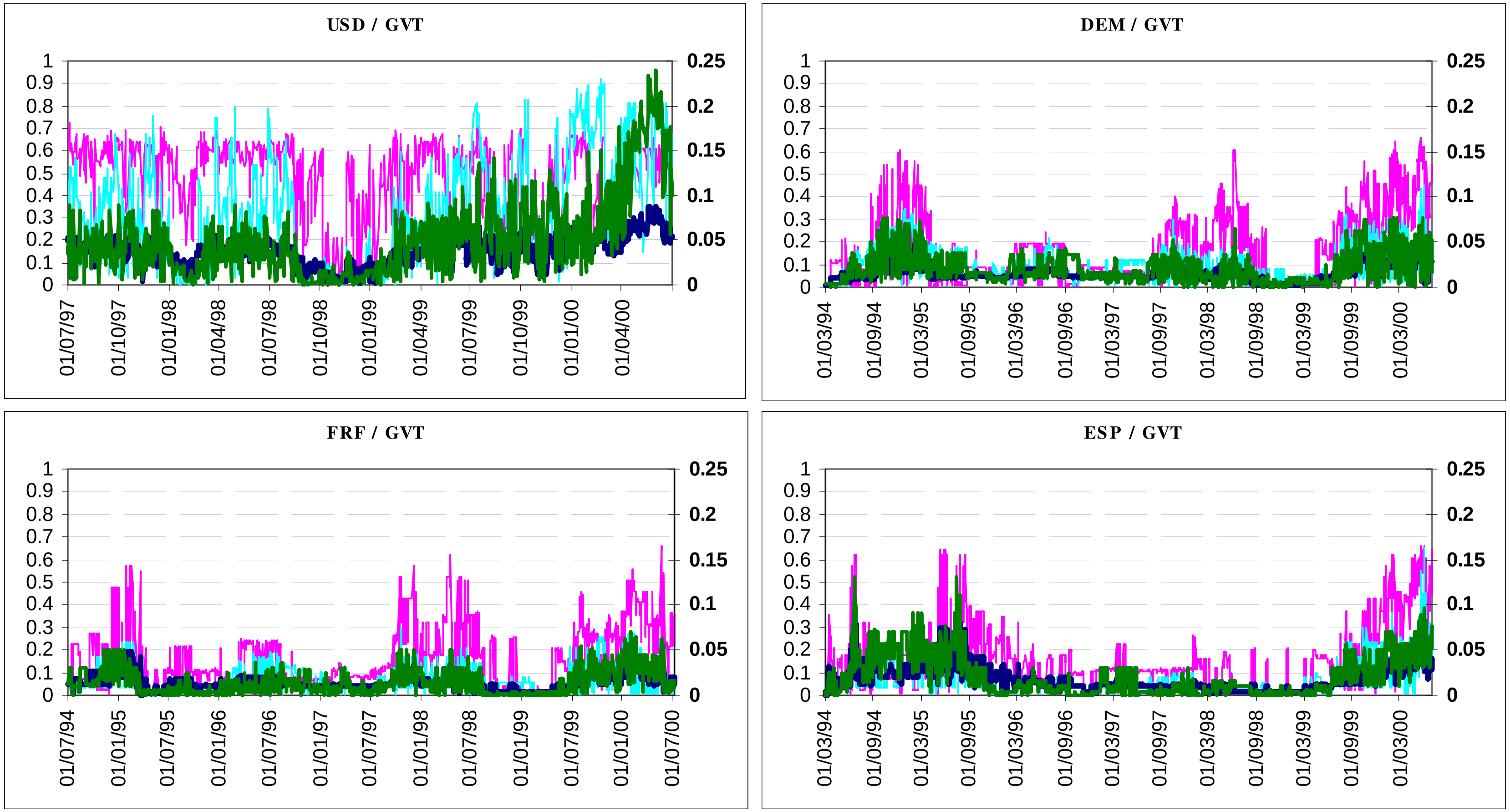


Figure 5 b - Estimates of the parameters of the DK1 model
(thin lines, left axis: b and ν, bold lines, right axis: a and μ)

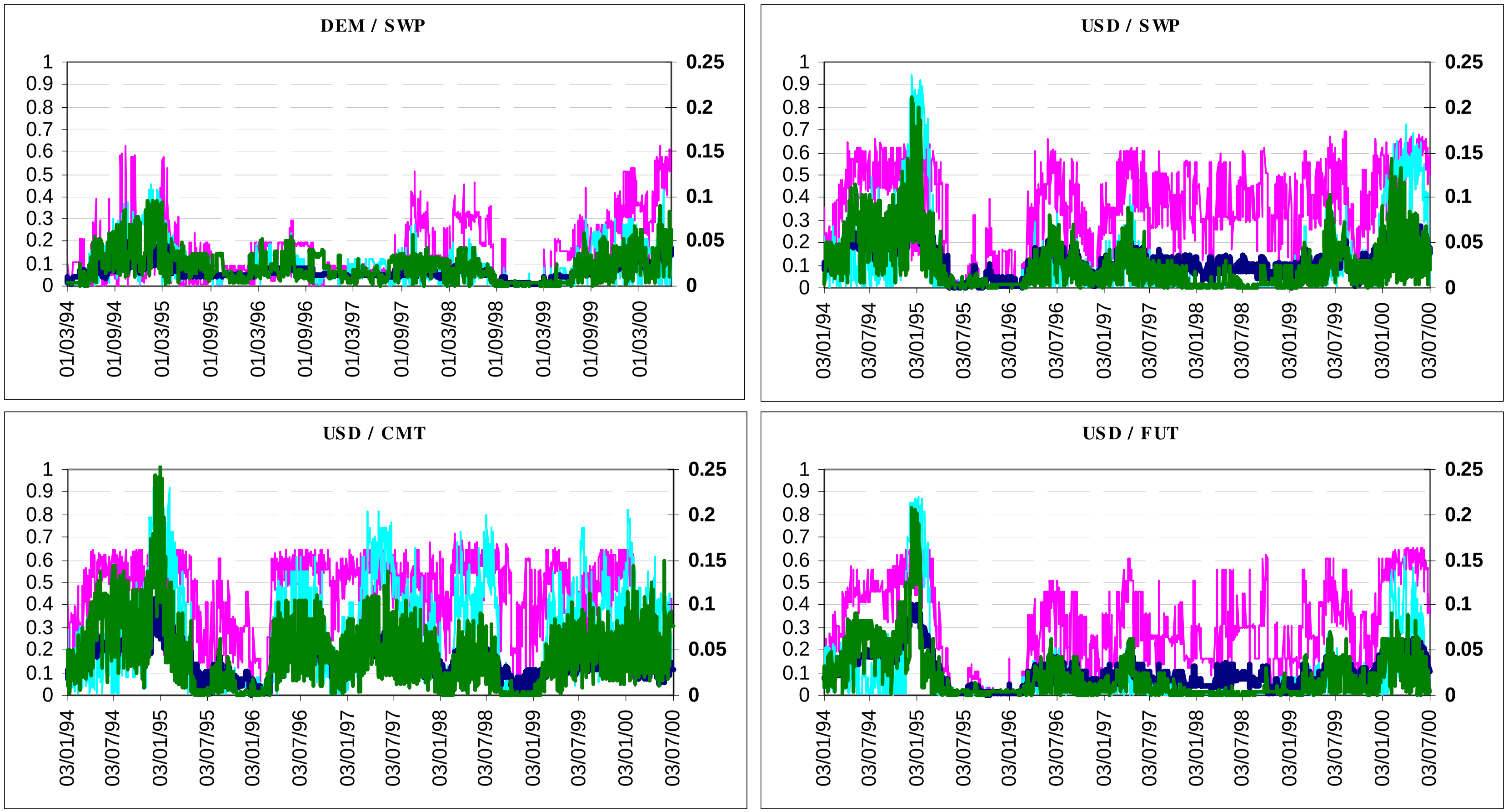
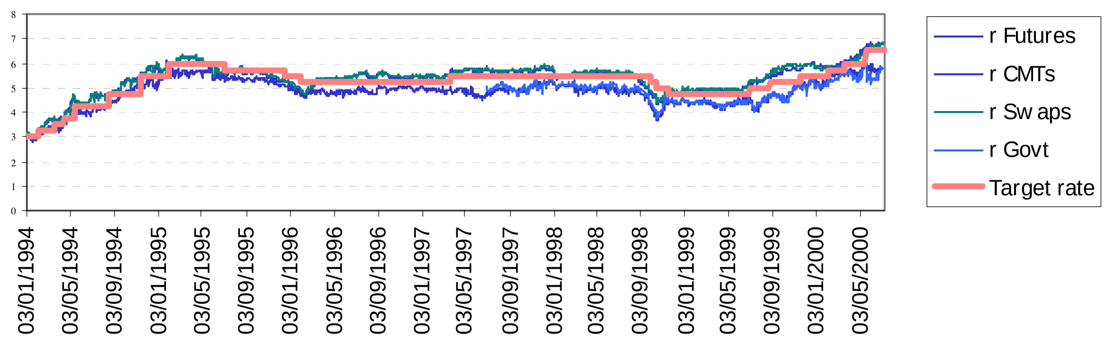


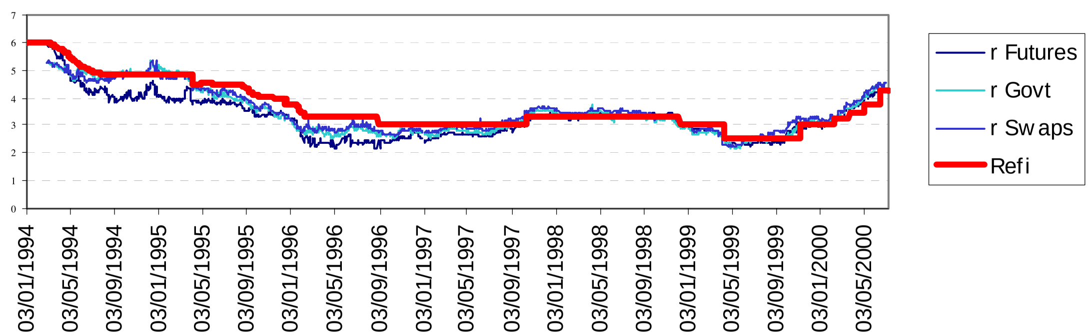
Figure 6 - Implied values of the short-term rate versus policy rate

(thin lines, implied values for r; bold lines, policy rates)

US



Germany



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